

## DESCENT CONSTRUCTION FOR GSPIN GROUPS

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ABSTRACT. In this paper we provide an extension of the theory of *descent* of Ginzburg-Rallis-Soudry to the context of essentially self-dual representations, that is representations which are isomorphic to the twist of their own contragredient by some Hecke character. Our theory supplements the recent work of Asgari-Shahidi on the functorial lift from (split and quasisplit forms of)  $GSpin_{2n}$  to  $GL_{2n}$ .

## 1. INTRODUCTION

A fundamental problem in modern number theory is the determination of the spectral decomposition of the right regular representation of  $G(\mathbb{A})$  on the function space  $L^2(G(F)\backslash G(\mathbb{A}))$ , where  $G$  is a reductive  $F$ -group and  $\mathbb{A}$  is the adele ring of the global field  $F$ . The constituents of the decomposition are special representations of  $G(\mathbb{A})$  that are called automorphic representations and encode analytic, geometric and number theoretic information. The *Langlands program*, a vast generalization of class-field theory, offers a description of these constituents in terms of certain homomorphisms from the conjectural Langlands group  $L_F$  to a complex group  ${}^L G$ , the  $L$ -group associated with  $G$ . Schematically, one expects a meaningful correspondence between the two sets

$$\{\pi \text{ automorphic representation of } G(\mathbb{A})\} \longleftrightarrow \{\text{admissible } \phi : L_F \rightarrow {}^L G\}$$

and sometimes it is possible to make this expectation exact.

**1.1. Functoriality.** One approach towards making the Langlands philosophy an exact prediction is via a conjectural relation called *functorial transfer*. Namely, whenever we have a homomorphism  $r : {}^L H \rightarrow {}^L G$  one should expect, in view of the correspondence above, a relation between automorphic representations on  $H(\mathbb{A})$  and those on  $G(\mathbb{A})$ . To give a more precise description of the expected correspondence we recall that an automorphic representation  $\pi$  admits a presentation as  $\pi = \otimes \pi_v$  where each  $\pi_v$  is an irreducible admissible representation of  $G(F_v)$ . Furthermore for almost all places  $v$  of  $F$ , the subgroup  $K_v = G(\mathfrak{o}_v) \subset G(F_v)$  is a maximal compact subgroup and the representations  $\pi_v$  admit a fixed vector with respect to  $K_v$ . Representations  $\rho$  of  $G(F_v)$  with this property are called *unramified* and to each one may attach a semisimple conjugacy class  $S(\rho) \subset {}^L G$  called the Satake parameters of  $\rho$ . For an automorphic representation  $\pi = \otimes \pi_v$  we denote  $S_v(\pi) = S(\pi_v)$ . The ordered collection  $(S_v(\pi))$  of Satake parameters serves as a fingerprint of the automorphic representation  $\pi$  and is used to compare automorphic representations. Given  $r : {}^L H \rightarrow {}^L G$  as above and  $\pi = \otimes \pi_v$  an automorphic representations on  $G(\mathbb{A})$  the *functoriality conjecture* predicts the existence of an automorphic representations  $\sigma$  on  $H(\mathbb{A})$  which is a weak lift of  $\pi$ , meaning

$$r(S_v(\pi)) = S_v(\sigma)$$

holds for almost all places  $v$ .

The problem of establishing the existence of functorial lifts relating automorphic representations on different groups attracted a lot of research and several methods have been used to establish such a relation. Roughly speaking, one can classify the attempts to establish the functoriality conjecture into three major approaches: the method of the trace formula, the method of the converse theorem supplemented with a theory of  $L$ -functions, and various methods of explicit constructions of automorphic representations. In recent years, all these methods have been developed and refined and many new instances of functoriality were established.

In the present paper we focus on the method of explicit constructions by the method of descent. We extend the descent method of Ginzburg, Rallis, and Soudry to  $\mathrm{GSpin}$  groups and are able to obtain information about the image of the functorial lift constructed in [Asg-Sha1] and [Asg-Sha3]. Below we provide more information.

**1.2. Self dual representations of  $GL_n(\mathbb{A})$  and the descent method.** Given a classical group  $H$  (e.g.  $Sp(2n)$ ,  $SO(2n+1)$ ,  $SO(2n)$ ) the dual group  ${}^L H$  is a classical group as well (e.g.  $SO(2n+1, \mathbb{C})$ ,  $Sp(2n, \mathbb{C})$ ,  $SO(2n, \mathbb{C})$ ) and naturally embedded in a general linear group. Now this standard embedding  ${}^L H \rightarrow GL_N(\mathbb{C}) = {}^L GL_N$  should conjecturally yield a functorial lift from each automorphic representation  $\sigma$  of  $H(\mathbb{A})$  to an automorphic representation  $\tau$  of  $GL(N, \mathbb{A})$ . Analyzing the Satake parameters of these lifts one expect  $\tau$  to be self dual, that is  $\tilde{\tau} \cong \tau$ .

In [C-K-PS-S2] the method the of converse theorem is used to show the existence of a weak functorial lift from generic cuspidal automorphic representations of classical groups to automorphic representations of the general linear group.

The theory of descent for self-dual cuspidal representations of the general linear group  $GL_n(\mathbb{A})$  was developed in a sequence of remarkable works [GRS1]-[GRS5]. The definitive account of this work is now available in [GRS7]. In these works the authors showed, among other things, that every self-dual irreducible cuspidal automorphic representation  $\pi$  of  $GL_n(\mathbb{A})$  is a weak functorial lift from a cuspidal representation of some classical group. Moreover, this was accomplished by explicitly constructing a space  $\sigma(\pi)$  of cuspidal automorphic functions which weakly lifts to a  $\pi$ . Thus the construction of  $\sigma(\pi)$  from  $\pi$  reverses the *lifting* constructed in [C-K-PS-S2], and for this reason is known as the “descent.” The precise classical group on which  $\sigma(\pi)$  is defined is governed by poles of the symmetric and exterior square  $L$  functions of  $\pi$ . For example, if  $L(\pi, \wedge^2, s)$  has a pole at  $s = 1$  then  $\sigma(\pi)$  is defined on  $SO_{2n+1}(\mathbb{A})$ .

Among other things, the authors of [GRS4] were able to use their descent construction to describe the image of the functorial lift of [C-K-PS-S1].

Thus, the conjunction of the descent method and associated integral representations with the method of the converse theorem provides a very detailed description of the image of functoriality corresponding to the standard embedding of  ${}^L G \rightarrow GL_N(\mathbb{C})$  with  $G$  a classical group. For an excellent survey we refer the reader to [So1].

The recent proof of the fundamental Lemma (and its twisted versions) by [Ngo], and the work of Arthur on the trace formula and its application to functoriality is expected to lead to a complete proof of the functoriality conjecture for the case when  $r$  is the standard embedding of any classical group. We refer the reader to [Ar] for the expected results. The method of the trace formula has the advantage that it is not restricted to generic representations of classical groups. On the other hand, it is indirect and one of the advantage of the descent method is that it provides an explicit realization of the automorphic representation constructed as a space of functions.

**1.3. Beyond classical.** Observe that each classical group is contained in a corresponding similitude group. One is naturally led to consider the case when  $r$  is the standard embedding of a similitude group. However, the dual group of a similitude classical group is not another similitude classical group, but rather a new type of group called a GSpin group. Thus in order to consider the case when  $r$  is the standard embedding of a similitude classical group, one must deal with automorphic forms defined on GSpin groups. Recently, Asgari and Shahidi proved in [Asg-Sha1] and [Asg-Sha3] the existence of a weak functorial lifting from each quasisplit GSpin group to the appropriate general linear group. The representations obtained in this fashion are essentially self dual. Moreover, in the special case of  $GSp(4)$  they were able to show in [Asg-Sha2] that this weak functorial lift is in fact strong in an appropriate sense.

**1.4. Descent construction for essentially self dual representations.** In this paper we extend the descent method of Ginzburg, Rallis, and Soudry to GSpin groups. As a bonus, for  $n \geq 2$  we can provide a “lower bound” on the image of each of the functorial lifts constructed by Asgari and Shahidi. For the case of  $GSpin_5 = GSp_4$ , these results were obtained by another method in [Gan-Tak]. Since this work was completed, Asgari and Shahidi have shown that this “lower bound” is, in fact, the full image [Asg-Sha3].

We now describe a few properties of GSpin groups. For more details see section 4. There is a unique quasisplit form of  $GSpin_{2n+1}$ , while quasisplit forms of  $GSpin_{2n}$  are in natural one-to-one correspondence with quadratic characters  $\chi : \mathbb{A}^\times/F^\times \rightarrow \pm 1$ . We will denote the corresponding group by  $GSpin_{2n}^\chi$ . We also denote by  $F[\chi]/F$  the quadratic extension corresponding to  $\chi$ .

We then obtain maps

$$r : {}^L(GSpin_{2n}^\chi) = GSO_{2n}(\mathbb{C}) \rtimes \text{Gal}(F[\chi]/F) \rightarrow GL_{2n}(\mathbb{C}) = {}^LGL_{2n}.$$

$$r : {}^L(GSpin_{2n+1}) = GSO_{2n}(\mathbb{C}) \rightarrow GL_{2n}(\mathbb{C}) = {}^LGL_{2n}.$$

sending the nontrivial element of  $\text{Gal}(F[\chi]/F)$  to

$$\begin{pmatrix} I_{n-1} & & & \\ & & 1 & \\ & & & \\ & 1 & & \\ & & & I_{n-1} \end{pmatrix}.$$

In each of the theorems below the notion of weak lift is defined relative to the appropriate map  $r$  as above.

To discern the form of  $GSpin_{2n}$  to which a given representation  $\tau$  will descend, we observe that  $\tau \cong \tilde{\tau} \otimes \omega$  implies  $\omega_\tau^2 = \omega^{2n}$ . Here  $\omega_\tau$  denotes the central character of  $\tau$ . Hence  $\omega_\tau/\omega^n$  is some quadratic character  $\chi$ .

Let us now state the main results of the present paper. For simplicity of exposition we consider the odd case and the even case separately. The results are analogous to the ones of [GRS4] describing endoscopic lifts from orthogonal and symplectic groups. In our formulation we use the notion of  $\bar{\omega}$ -symplectic and  $\bar{\omega}$ -orthogonal representations which is introduced in 2.2

We begin with main result of part 1 of this work.

**Theorem (MAIN THEOREM: ODD CASE).** *For  $r \in \mathbb{N}$ , take  $\tau_1, \dots, \tau_r$  to be irreducible unitary automorphic cuspidal representations of  $GL_{2n_1}(\mathbb{A}), \dots, GL_{2n_r}(\mathbb{A})$ , respectively, and let  $\tau = \tau_1 \boxplus \dots \boxplus \tau_r$  be the isobaric sum (see section 2.4). Let  $\omega$  denote a Hecke character. Suppose that*

- $\tau_i$  is  $\bar{\omega}$ -symplectic for each  $i$ , and
- $\tau_i \cong \tau_j \Rightarrow i = j$ .

*Then there exists an irreducible generic cuspidal automorphic representation  $\sigma$  of  $GSpin_{2n+1}(\mathbb{A})$  such that*

- $\sigma$  weakly lifts to  $\tau$ , and
- the central character of  $\sigma$  is  $\omega$ .

Next, we formulate the main result of part 2.

**Theorem (MAIN THEOREM: EVEN CASE).** *For  $r \in \mathbb{N}$ , take  $\tau_1, \dots, \tau_r$  to be irreducible unitary automorphic cuspidal representations of  $GL_{2n_1}(\mathbb{A}), \dots, GL_{2n_r}(\mathbb{A})$ , respectively, and let  $\tau = \tau_1 \boxplus \dots \boxplus \tau_r$  be the isobaric sum (see section 2.4). Let  $n = n_1 + \dots + n_r$ , and assume that  $n \geq 2$ . Let  $\omega$  denote a Hecke character, which is not the square of another Hecke character. Suppose that*

- $\tau_i$  is  $\omega^{-1}$ -orthogonal for each  $i$ , and
- $\tau_i \cong \tau_j \Rightarrow i = j$ .

*For each  $i$ , let  $\chi_i = \omega_{\tau_i}/\omega^{n_i}$  (which is quadratic), and let  $\chi = \prod_{i=1}^r \chi_i$ . Then there exists an irreducible generic cuspidal automorphic representation  $\sigma$  of  $GSpin_{2n}^{\chi}(\mathbb{A})$  such that*

- $\sigma$  weakly lifts to  $\tau$ , and
- the central character  $\omega_{\sigma}$  of  $\sigma$  is  $\omega$ .

**1.5. Applications.** In a series of works, Asgari and Shahidi studied the functorial lifts from Spinor groups to the general linear group. We first formulate their result for the odd case. The next theorem is Theorem 1.1 (p. 140) of [Asg-Sha1].

**Theorem 1.5.1.** *Let  $\pi$  be an irreducible globally generic cuspidal automorphic representation of  $GSpin_{2n+1}(\mathbb{A})$ . Then there exists an automorphic representation  $\tau = AS(\pi)$  of  $GL_{2n}(\mathbb{A})$  which is a weak lift of  $\pi$  with respect to the map  $r$  of (2.1.1). Furthermore, denoting by  $\omega_{\pi}, \omega_{\tau}$  the central characters of  $\pi, \tau$  we have  $\omega_{\tau} = \omega_{\pi}^n$ . Finally,  $AS(\pi)$  and  $\widetilde{AS(\pi)} \otimes \omega_{\pi}$  are nearly equivalent.*

The present work allows a description of the image of the lifting constructed by Asgari-Shahidi. Thus we have the following

**Corollary 1.1.** *The image of the functorial lift  $AS$  (see Theorem 1.5.1) contains the set of all representations  $\tau_1 \boxplus \dots \boxplus \tau_r$  such that*

- $\tau_i \cong \tau_j \Rightarrow i = j$ ,
- there is a Hecke character  $\omega$  such that  $\tau_i$  is  $\bar{\omega}$ -symplectic for each  $i$ .

The above result of the present work was used in [Asg-Rag].

**Remark 1.5.2.** *Since this work was completed, it has been shown in [Asg-Sha3] that the image of the functorial lift  $AS$  is actually equal to the set described above.*

For the even case, the results are similar, though the sequence of events is somewhat different. The next theorem is Theorem 5.16 of [Asg-Sha3].

**Theorem 1.5.3.** *Let  $\pi$  be an irreducible globally generic cuspidal automorphic representation of  $GSpin_{2n}^\chi(\mathbb{A})$ . Then there exists an automorphic representation  $\tau = AS_\chi(\pi)$  of  $GL_{2n}(\mathbb{A})$  which is a weak lift of  $\pi$  with respect to the map  $r$  of (12.0.6). Furthermore, denoting by  $\omega_\pi, \omega_\tau$  the central characters of  $\pi, \tau$  we have  $\omega_\tau = \omega_\pi^n \chi$ . Finally,  $AS_\chi(\pi)$  and  $\widetilde{AS_\chi(\pi)} \otimes \omega_\pi$  are nearly equivalent.*

The contribution of this work is a lower bound for the image of this functorial lift.

**Corollary 1.2.** *The image of the functorial lift  $AS_\chi$  described in Theorem 1.5.3 contains the set of all representations  $\tau_1 \boxplus \cdots \boxplus \tau_r$  such that*

- $\tau_i \cong \tau_j \Rightarrow i = j$ ,
- there is a Hecke character  $\omega$  such that  $\tau_i$  is  $\omega^{-1}$ -orthogonal for each  $i$ .

As with the odd case, it is shown in [Asg-Sha3] that the image of the functorial lift  $AS_\chi$  is actually equal to the set described above.

**1.6. The descent construction and the structure of the argument.** We give here a brief outline of the ingredients one use to prove the main theorems. We refer the reader to [HS] where we have provided a detailed description for the scheme of the argument for the even case.

The construction of the descent representations relies on the notion of Fourier coefficient, as defined in [GRS8], [G] (cf. also the “Gelfand-Graev” coefficients of [Sol]). For purposes of presenting certain of the global arguments, it seems convenient to embed these Fourier coefficients into a slightly larger family of functionals, which we shall refer to as “unipotent periods.”

Suppose that  $U$  is a unipotent subgroup of  $G$  and  $\psi$  is a character of  $U(F) \backslash U(\mathbb{A})$ . We define the corresponding *unipotent period* to be the map from smooth, left  $U(F)$ -invariant functions on  $G(\mathbb{A})$  to smooth, left  $(U(\mathbb{A}), \psi)$ -equivariant functions, given by

$$\varphi \mapsto \varphi^{(U, \psi)}$$

where

$$\varphi^{(U, \psi)}(g) := \int_{U(F) \backslash U(\mathbb{A})} \varphi(ug) \psi(u) du.$$

Let  $\mathcal{S}$  be a set of unipotent periods. We will say that another unipotent period  $(U, \psi)$  is *spanned* by  $\mathcal{S}$  if the implication

$$(1.6.1) \quad \left( \varphi^{(N, \vartheta)} \equiv 0 \quad \forall (N, \vartheta) \in \mathcal{S} \right) \implies \varphi^{(U, \psi)} \equiv 0$$

is valid for any automorphic function  $\varphi$ .

For simplicity of the exposition we assume that we are trying to construct a descent for an  $\bar{\omega}$ -orthogonal cuspidal representation,  $\tau$  of the group  $GL_{2n}(\mathbb{A})$ . Recall that the central character of  $\tau$  is equal to  $\omega^n \chi_\tau$  for some quadratic character  $\chi_\tau$ .

We can conveniently describe the method in the following steps:

- (1) Construction of a family of descent representations of  $GSpin_{4n-2\ell}^\chi(\mathbb{A})$  for  $\ell \geq n$ , and  $\chi$  any quadratic character.
- (2) Vanishing of the descent representations for all  $\ell > n$  and all  $\chi \neq \chi_\tau$ .
- (3) Cuspidality and genericity (hence nonvanishing) of the descent representation of  $GSpin_{2n}^{\chi_\tau}(\mathbb{A})$ .
- (4) Matching of spectral parameters at unramified places.

For the construction of the descent representations, we begin, in section 15, with an Eisenstein series on the group  $GSpin_{4n+1}$  that is induced from a Levi  $M$  isomorphic to  $GL_{2n} \times GL_1$ . The representations  $\tau$ , properly twisted, is used as the inducing data and a pole of  $L(s, \tau, \text{sym}^2 \times \bar{\omega})$  allows us to construct a residue representation  $\mathcal{E}_{-1}(\tau, \omega)$  of  $GSpin_{4n+1}(\mathbb{A})$ . Next, for each  $\ell$  and  $\chi$  we give in section 16.1 an embedding of  $GSpin_{4n-2\ell}^\chi$  into  $GSpin_{4n+1}$ , and construct, using formation of a Fourier coefficient, a space of functions on this subgroup of  $GSpin_{4n+1}$ .



Let  $DC_\omega^\chi(\tau)$  denote the space of functions on the quasisplit form  $GSpin_{2n}^\chi$  of  $GSpin_{2n}$  corresponding to the character  $\chi$ . Then, using a local argument involving twisted Jacquet modules which is given in section 20, we prove that  $DC_\omega^\chi(\tau)$  is zero, except when  $\chi = \omega_\tau/\omega^n$ . A similar argument is used to prove the vanishing of the “deeper descents,” defined on  $GSpin_{4n-2\ell}^\chi$  for all  $\ell > n$  and all  $\chi$ .

If  $\chi = \chi_\tau$ , then we show that  $DC_\omega^\chi(\tau)$  is nonzero, even generic, and all of the functions in it are cuspidal. This is based on relations of unipotent periods of the type given in (1.6.1), which are proved in section 21. It follows that  $DC_\omega^\chi(\tau) = \oplus \sigma_i$  decomposes as a direct sum of irreducible automorphic cuspidal representations  $\sigma_i$  of  $GSpin_{2n}^\chi$ . We then show, using again a local argument in section 20, that each of these irreducible constituents  $\sigma_i$  lifts weakly to  $\tau$  by the functorial lifting  $AS_{\chi_\tau}$ .

To treat the general case, the representation  $\tau$  may be an isobaric sum of several cuspidal representations  $\tau_1, \dots, \tau_r$  and one considers a more complicated Eisenstein series. The main differences in the construction are that the residue is a multi-residue, and the notation is more complicated. But the idea of the construction is similar. See sections 12 - 21.

The  $\bar{\omega}$ -symplectic case is similar, and is covered in sections 6 to 11.

**1.7. Structure of the paper.** In constructing this paper, we have tried to emphasize self-containment and readability over efficiency. Therefore we write out in detail many arguments on Eisenstein series which may be regarded as standard and well known to the experts. Also, we have, written up the odd and even cases separately, and in full detail, even though there is a great overlap between the two. At each step, the statements and proofs in the odd and even cases are variations of one another. However, we felt it was desirable to give complete details for both cases, and that would be easier to follow one argument at a time than two variations developed in parallel. The only significant deviation from this principle is the treatment of Eisenstein series, for which we have included a complete and detailed proof only in the even case. The proof in the odd case is omitted because it is virtually identical, *and* because both proofs consist mainly of specializing general results on Eisenstein series to a particular case.

If one is interested in reading only the odd case, and is willing to take as given some standard facts about Eisenstein series, this case is contained in parts 1 and 2. For a self-contained treatment of the even case, one should read parts 1 and 3.

We now quickly summarize the logic of the paper. In section 2 we formulate the main result in the odd case. In sections 3 and 4 we set up some general notions, and introduce  $GSpin$  groups. We collect here information about root datum, Weyl group, and unipotent subgroups. Section 5 gives some general material concerning unipotent periods which may be useful in other contexts. With these general matters completed, we begin the detailed treatment of the odd case. Section 6 contains the precise statement of the main theorem in the odd case, and a little bit of additional notation. In section 7 we consider the Satake parameters of essentially self-dual unramified representations on  $GL_{2n}$ . We show that these are always parameters associated with representations of smooth representations of  $GSpin$  groups. This is the local underpinning of the descent construction. In section 8 we introduce a specific family of Eisenstein series, and formulate their main properties, in Theorem 8.0.12. In section 9 we prove the main result for the odd case. Sections 10 and 11 are appendices to the odd case, containing proofs of various technical results. In the remaining sections we repeat all of these steps for the “even” case (descent from  $GL_{2n}$  to some quasisplit form of  $GSpin_{2n}$ ).

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Without David Ginzburg and David Soudry's many careful and patient explanations of the "classical" case— $\omega = 1$ —this work would not have been completed. It is important to point out that not all of the arguments shown to us have appeared in print. We mention in particular the computation of Jacquet modules in Appendix V, and the nonvanishing argument in Appendix VI. Nevertheless, in each case the specialization of our arguments to the case  $\omega = 1$  may be correctly attributed to Ginzburg, Rallis, Soudry (with any errors or stylistic blemishes introduced being our own responsibility).

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## Part 1. General matters

### 2. SOME NOTIONS RELATED TO LANGLANDS FUNCTORIALITY

To formulate the main result of the paper we need the terminology of weak lift, the notion of essentially self dual representations and the notion of isobaric sums. We review them here briefly but the reader can go directly to 6.

**2.1. Weak Lift.** Let  $G$  and  $H$  be reductive groups defined over a number field  $F$ . Let  $\pi \cong \otimes'_v \pi_v$  be an automorphic representation of the group  $G(\mathbb{A})$ , where  $\mathbb{A}$  is the ring of adeles of  $F$ .

At almost all places  $v$  the smooth representation  $\pi_v$  of  $G(F_v)$  is unramified. The semi-simple conjugacy class in the  $L$ -group  ${}^L G$  associated by the Satake isomorphism to  $\pi_v$  will be denoted  $t_{\pi_v}$ .

We say that an automorphic representation  $\sigma$  of  $G(\mathbb{A})$  is a *weak lift* of the automorphic representation  $\tau \cong \otimes'_v \tau_v$  of  $H(\mathbb{A})$  with respect to a map  $r : {}^L H \rightarrow {}^L G$  if for almost all places,  $r(t_{\sigma_v}) \subset t_{\tau_v}$ .

We now specialize to the case  $H = GSpin_{2n+1}$  (for definition and properties see [Asg] and section 4)  $G = GL_{2n}$  and the inclusion

$$(2.1.1) \quad r : {}^L H = {}^L(GSpin_{2n+1}) = GSp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}) = {}^L GL_{2n} = {}^L G.$$

**2.2. Essentially self-dual:  $\eta$ -orthogonal and  $\eta$ -symplectic.** To formulate our main result we introduce the notion of  $\eta$ -orthogonal and  $\eta$ -symplectic representations. Let  $\tau$  be an irreducible automorphic cuspidal representation of  $GL_{2n}$ , and let  $\eta$  be a Hecke character. Recall that the theory of self dual representations is well developed. Suppose that  $\tau$  is *essentially self-dual*, i.e. that

$$\tilde{\tau} = \tau \otimes \eta$$

Where  $\tilde{\tau}$  is the contragredient of  $\tau$  and  $\eta$  is some Hecke character.

**Definition 2.2.1.** Let  $\tau$  be an irreducible automorphic cuspidal representation of  $GL_{2n}$  and  $\eta$  a Hecke character. Consider the (exterior) tensor product representation  $\tau \boxtimes \eta$  of the group  $GL_{2n}(\mathbb{A}) \times GL_1(\mathbb{A})$ . The finite Galois form of the  $L$  group of  $GL_{2n}(\mathbb{A}) \times GL_1(\mathbb{A})$  is the group  $GL_{2n}(\mathbb{C}) \times GL_1(\mathbb{C})$ .

We write  $L(s, \tau, \wedge^2 \times \eta)$  ( resp.  $L(s, \tau, \text{sym}^2 \times \eta)$ ) for the  $L$  function attached to  $\tau \boxtimes \eta$  and the representation of  $GL_{2n}(\mathbb{C}) \times GL_1(\mathbb{C})$  where  $GL_{2n}(\mathbb{C})$  acts by the exterior (resp. symmetric) square representation and  $GL_1(\mathbb{C})$  acts by scalar multiplication.



**Lemma 2.1.** *Let  $\tau$  be an irreducible automorphic cuspidal representation of  $GL_{2n}$  and  $\eta$  a Hecke character. Consider the  $L$  functions  $L(s, \tau, \wedge^2 \times \eta)$  and  $L(s, \tau, \text{sym}^2 \times \eta)$ .*

- (1) *If  $\tilde{\tau}$  is isomorphic to  $\tau \otimes \eta$  for some Hecke character  $\eta$  then exactly one of these  $L$ -functions has a simple pole at  $s = 1$  while the other is holomorphic and nonvanishing.*
- (2) *if  $\tilde{\tau}$  is not isomorphic to  $\tau \otimes \eta$  then both functions are holomorphic at  $s = 1$ .*

*Proof.* It follows from [MW2], the corollaire on p.667, that  $L(s, \tau \times \tau \otimes \eta)$  has a simple pole at  $s = 1$  if  $\tau \otimes \eta = \tilde{\tau}$  and is holomorphic at  $s = 1$  otherwise.

Now,  $L(s, \tau \times \tau \otimes \eta)$  is the Langlands  $L$  function of on  $M_{2n \times 2n}(\mathbb{C})$  in which  $GL_{2n}(\mathbb{C})$  acts by  $g \cdot X = gX {}^t g$  and  $GL_1(\mathbb{C})$  acts by scalar multiplication. But this representation is reducible, decomposing into the subspaces of skew-symmetric and symmetric matrices. Thus we have

$$L(s, \tau \times \tau \otimes \eta) = L(s, \tau, \wedge^2 \times \eta) L(s, \tau, \text{sym}^2 \times \eta).$$

It is proved in [Sha6], theorem 4.1, p. 321, that neither  $L(s, \tau, \wedge^2 \times \eta)$  nor  $L(s, \tau, \text{sym}^2 \times \eta)$  vanishes on the line  $\text{Re}(s) = 1$ . The result follows □

We are ready for a definition

- Definition 2.2.**
- (1) *We will say that  $\tau$  is  $\eta$ -symplectic in case  $L(s, \tau, \wedge^2 \times \eta)$  has a pole at  $s = 1$ .*
  - (2) *We will say that  $\tau$  is  $\eta$ -orthogonal in case  $L(s, \tau, \text{sym}^2 \times \eta)$  has a pole at  $s = 1$ .*

Thus, if  $\tau$  is essentially self dual, there exists a Hecke character  $\eta$  such that  $\tau$  is either  $\eta$ -symplectic or  $\eta$ -orthogonal.

- Remarks 2.2.2.**
- (1) *Above we have defined “ $\eta$ -symplectic” and “ $\eta$ -orthogonal” using completed  $L$  functions. We could also have used partial  $L$  functions. The notions obtained are equivalent. (See item (ii): near the end of the proof of proposition 18.0.4, and its proof.)*
  - (2) *There is another natural notion of “orthogonal/symplectic representation.” Specifically, one could say that an automorphic representation is orthogonal/symplectic if the space it acts on supports an invariant symmetric/skew-symmetric form. The two notions appear to be related, but do not coincide. See [PraRam].*
  - (3) *There is a third approach to defining a local factor for  $L(s, \tau, \wedge^2 \times \eta)$ , which is to apply the “gcd” construction described in [Gel-Sha] section I.1.6, p. 17, to the integrals in [Ja-Sh1]. As far as we know this is not written down anywhere.*
  - (4) *An integral representation for  $L(s, \tau, \text{sym}^2)$  was given in [BG]. The problem of extending this to  $L(s, \tau, \text{sym}^2 \times \eta)$  has been solved by Banks [Banks1, Banks2] in the case  $n = 3$  and Takeda [Tak] for general  $n$ .*

**2.3. Isobaric sums.** We recall the following result.

- Theorem 2.3.**
- (1) *every irreducible automorphic representation of  $GL_n(\mathbb{A})$  is isomorphic to a subquotient of  $\text{Ind}_{P(\mathbb{A})}^{GL_n(\mathbb{A})} \tau_1 |\det_1|^{s_1} \otimes \cdots \otimes \tau_r |\det_r|^{s_r}$  for some real numbers  $s_1, \dots, s_r$  and irreducible unitary automorphic cuspidal representations  $\tau_1, \dots, \tau_r$  of  $GL_{n_1}(\mathbb{A}), \dots, GL_{n_r}(\mathbb{A})$  respectively, such that  $n_1 + \cdots + n_r = n$ . Here  $P$  is the standard parabolic of  $GL_n$  corresponding to the ordered partition  $(n_1, \dots, n_r)$  of  $n$ .*
  - (2) *In the case when  $s_i = 0$  for all  $i$ , this induced representation is irreducible.*
  - (3) *The representations obtained when  $s_i = 0$  for all  $i$  by numbering a given set of cuspidal representations in different ways are isomorphic.*
  - (4) *If two induced representations as in (1), with  $s_i = 0$  for all  $i$ , are isomorphic, then they are obtained from two numberings of the same set of cuspidal representations*

*Proof.* (1) This follows from proposition 2 of [L3].

(2) This follows from the irreducibility of all the local induced representations, which is Theorem 3.2 of [Ja].

(3) This follows from the fact that the standard intertwining operator between them does not vanish, which follows from [MW1], II.1.8 (meromorphically continued in IV.1.9(e)), and IV.1.10(b). In IV.3.12 these elements are combined to prove that the intertwining operator does not have a pole. The proof that it does not have a zero is an easy adaptation.

(4) This follows from [Ja-Sh3], Theorem 4.4, p.809. □

**Definition 2.4.** Let  $\tau_1, \dots, \tau_r$  be irreducible unitary cuspidal representations of  $GL_{n_1}(\mathbb{A}), \dots, GL_{n_r}(\mathbb{A})$ . The isobaric sum of  $\tau_1 \boxplus \dots \boxplus \tau_r$  of these representations is the irreducible unitary representation  $\tau$  of  $GL_n(\mathbb{A})$  (where  $n = n_1 + \dots + n_r$ ) given in (1) above, with  $s_i = 0$  for all  $i$ .

**Remark 2.3.1.** A more general notion of “isobaric representation” was introduced in [L4], but we don’t need it.

### 3. NOTATION

**3.1. General.** Throughout most of the paper,  $F$  will denote a number field.

We denote by  $J$  the matrix, of any size, with ones on the diagonal running from upper right to lower left, and by  $J'$  the matrix  $\begin{pmatrix} & J \\ -J & \end{pmatrix}$  of any even size. We also employ the notation  ${}^t g$  for the transpose of  $g$  and  ${}^t g$  for the “other transpose”  $J {}^t g J$ . We employ the shorthand  $G(F \backslash \mathbb{A}) = G(F) \backslash G(\mathbb{A})$ , where  $G$  is any  $F$ -group.

We denote the Weyl group of the reductive group  $G$  by  $W_G$  or by  $W$ , when the meaning is clear from the context. Given  $\pi$  a local representation or an automorphic representation, we denote by  $\tilde{\pi}$  the contragredient of  $\pi$ , and by  $\omega_\pi$  its central character.

**3.2. Notions of “genericity”.** Let  $G$  be a quasisplit reductive group over the number field  $F$ , and  $U_{\max}$  a maximal unipotent subgroup. First let  $\psi_v$  be a generic character (cf. [Kim2], p. 147, and also [Shal], p.304) of  $U_{\max}(F_v)$  for some place  $v$  of  $F$ , and  $(\pi_v, V)$  a representation of  $G(F_v)$ . We say that  $\pi_v$  is  $\psi_v$ -generic if it supports a nontrivial  $\psi_v$ -Whittaker functional (i.e., a  $U_{\max}(F_v)$ -equivariant linear map  $V \rightarrow \mathbb{C}_{\psi_v}$ , which is continuous in an appropriate topology, see [Shal], p. 304. Here  $\mathbb{C}_{\psi_v}$  denotes the one-dimensional  $U_{\max}(F_v)$ -module with action via the character  $\psi_v$ .) Now let  $\pi \cong \otimes_v' \pi_v$  be a automorphic representation of  $G(\mathbb{A})$ , and let  $\psi = \prod_v \psi_v$  be a character of  $U_{\max}(F \backslash \mathbb{A})$ , by which we mean a character  $U_{\max}(\mathbb{A})$  which is trivial on  $U_{\max}(F)$ .

Ignoring topological considerations, it is easy to see that the space  $\text{Hom}_{U_{\max}(\mathbb{A})}(V_\pi, \mathbb{C}_\psi)$  is nontrivial iff each of the spaces  $\text{Hom}_{U_{\max}(F_v)}(V_{\pi_v}, \mathbb{C}_{\psi_v})$  is. However, it turns out that the more important issue is not whether there exists *some* nontrivial  $\psi$ -Whittaker functional, but whether the specific  $\psi$ -Whittaker functional given by

$$\varphi \mapsto \int_{U_{\max}(F \backslash \mathbb{A})} \varphi(u) \bar{\psi}(u) du$$

is nonvanishing. We refer to this Whittaker functional as the  $\psi$ -Whittaker *integral*. (See [Gel-So] for an example where the Whittaker integral vanishes, but a nonzero Whittaker functional exists.)

We would like to take this opportunity to draw attention to the subtle fact that there are two slightly different notions of global genericity for automorphic representations in common usage. The first states that a representation is globally  $\psi$ -generic if it supports a nonzero  $\psi$ -Whittaker integral. The second— which was the notion originally introduced in [PS]— requires that a cuspidal representation be *orthogonal to the kernel* of the  $\psi$ -Whittaker integral in  $L^2_{\text{cusp}}(G(F \backslash \mathbb{A}))$ , in order to be called “generic.” Clearly, the latter condition implies the former (except for the zero representation).

A nice feature of the stronger formulation is that the condition defines a subspace of  $L^2_{\text{cusp}}(G(F\backslash\mathbb{A}))$ , which one may term the  $\psi$ -generic spectrum. Furthermore, this subspace satisfies multiplicity one, even if  $L^2_{\text{cusp}}(G(F\backslash\mathbb{A}))$  does not. (Cf. [PS]) A nice feature of the weaker formulation is that it does not rely on the  $L^2$ -pairing, and hence no technicalities arise in applying the notion to non-cuspidal forms and representations.

Throughout most of this paper, we shall say that a representation “is  $\psi$ -generic” if it supports a nonzero  $\psi$ -Whittaker integral, and “is generic” if it satisfies this condition for *some*  $\psi$ . We shall say that a cuspidal representation is “in the  $\psi$ -generic spectrum” if it is orthogonal to the kernel of the  $\psi$ -Whittaker integral.

Let  $P_0 = N_G(U_{\max})$ . If  $P_0(F_v)$  permutes the characters of  $U(F_v)$  transitively, then we may refer to a representation as “generic” or “non-generic” without reference to a specific  $\psi_v$ , and without ambiguity. The same applies to both notions of global genericity, in the case when  $P_0(F)$  permutes the characters of  $U_{\max}(F\backslash\mathbb{A})$  transitively. This condition is satisfied by  $GL_n$  and  $GSpin_{2n+1}$ , but *not* by  $GSpin_{2n}$ .

#### 4. THE SPIN GROUPS $GSpin_m$ AND THEIR QUASISPLIT FORMS

We shall now define  $GSpin$  groups by introducing their root datum. They will be defined as the groups whose duals are the similitude classical groups  $GSp_{2n}(\mathbb{C}), GSO_{2n}(\mathbb{C})$ . Thus we begin by describing the based root data of these classical groups, but employ notation appropriate to the role these groups will eventually have as the duals of  $GSpin$  groups.

**4.1. Similitude groups  $GSp_{2n}, GSO_{2n}$ .** We first define the *similitude orthogonal and symplectic groups* to be

$$\begin{aligned} GO_m &= \{g \in GL_m : gJ^t g = \lambda(g)J \text{ for some } \lambda(g) \in \mathbb{G}_m\}, \\ GSp_{2n} &= \{g \in GL_{2n} : gJ'^t g = \lambda(g)J' \text{ for some } \lambda(g) \in \mathbb{G}_m\}. \end{aligned}$$

For each of these groups the map  $g \mapsto \lambda(g)$  is a rational character called the *similitude factor*.

**Remark 4.1.1.** *If  $m$  is odd then  $GO_m$  is in fact isomorphic to  $SO_m \times GL_1$ . This case will play no further role.*

The group  $GO_{2n}$  is disconnected; indeed let  $GSO_{2n}$  be the subgroup generated by  $SO_{2n}$  and  $\left\{ \begin{pmatrix} \lambda I_n & \\ & I_n \end{pmatrix} : \lambda \in \mathbb{G}_m \right\}$ . Then  $GSO_{2n}$  is connected and of index two in  $GO_{2n}$ .

The next lemma is straightforward.

**Lemma 4.1.** *Let*

$$T = \{t = \text{diag}(t_1, \dots, t_n, \lambda t_n^{-1}, \dots, \lambda t_1^{-1})\}.$$

*For  $i = 1$  to  $n$ , let  $e_i^*(t) = t_i$  and  $e_0^*(t) = \lambda$ .*

*Let  $X^\vee = \text{Hom}(T, \mathbb{G}_m)$  denote the character lattice of  $T$ . Then*

- *$T$  is a maximal torus in both groups  $GSp_{2n}$  and  $GSO_{2n}$*
- *$\{e_i^* : i = 0 \text{ to } n\}$  is a basis for  $X^\vee$ .*

Let  $X = \text{Hom}(\mathbb{G}_m, T)$  be the cocharacter lattice of  $T$ , and let  $\{e_i : i = 0 \text{ to } n\}$  be the basis of  $X$  dual to the basis  $\{e_i^* : i = 0 \text{ to } n\}$  of  $X^\vee$ .

Each similitude classical group has a Borel subgroup equal to the set of upper triangular matrices which are in it. In each case we employ this choice of Borel, and let  $\Delta^\vee$  denote the set of simple roots and  $\Delta$  the set of simple coroots.

**Lemma 4.2.** (1) *For  $GSp_{2n}$*

$$\begin{aligned} \Delta^\vee &= \{e_i^* - e_{i+1}^*, i = 1 \text{ to } n-1\} \cup \{2e_n^* - e_0^*\}. \\ \Delta &= \{e_i - e_{i+1}, i = 1 \text{ to } n-1\} \cup \{e_n\}. \end{aligned}$$

(2) For  $GSO_{2n}$

$$\begin{aligned}\Delta^\vee &= \{e_i^* - e_{i+1}^*, i = 1 \text{ to } n-1\} \cup \{e_{n-1}^* + e_n^* - e_0^*\}. \\ \Delta &= \{e_i - e_{i+1}, i = 1 \text{ to } n-1\} \cup \{e_{n-1} + e_n\}.\end{aligned}$$

**4.2. Definition of split  $GSpin$ .** We first deal with the *split* forms. By p.274 of [Spr]  $F$ -split connected reductive algebraic groups are classified by based root data. Thus the following makes sense.

**Definition 4.3.** (1) The group  $GSpin_{2n+1}$  is the  $F$ -split connected reductive algebraic group having based root datum dual to that of  $GSp_{2n}$ .

(2) The group  $GSpin_{2n}$  is the  $F$ -split connected reductive algebraic group having based root datum dual to that of  $GSO_{2n}$ .

We note that an alternative description of the same group is used in [Asg].

**Proposition 4.4.** The group  $GSpin_m$  is identical to the quotient of  $Spin_m \times GL_1$  as given in [Asg].

This is Proposition 2.4 on p. 678 of [Asg].

**4.3. Definition of quasi-split  $GSpin$ .** Now we turn to *quasi-split* forms. By the classification results in Chapter 16 of [Spr] (especially 16.3.2, 16.3.3 16.4.2), one finds that  $GSpin_{2n+1}$  is in fact the unique *quasi-split*  $F$ -group having based root datum dual to that of  $GSp_{2n}$ .

Furthermore, there is a 1-1 correspondence between quasi-split  $F$  groups  $G$  such that  ${}^L G^0 = GSO_{2n}(\mathbb{C})$  and homomorphisms from  $\text{Gal}(\bar{F}/F)$  to the group  $S$  of automorphisms of the lattice  $X(T)$  which permute the set  $\Delta$  of simple roots according to an automorphism of the Dynkin diagram.

**Lemma 4.5.** (1)  $S = \{1, \nu\}$  is of order two.

(2)  $\nu(e_{n-1} - e_n) = e_{n-1} + e_n, \nu(e_{n-1} + e_n) = e_{n-1} - e_n$  and  $\nu(\alpha) = \alpha$  if  $\alpha$  is any of the remaining simple roots.

$$(3) \nu(e_i) = \begin{cases} e_i & i \neq 0, n \\ -e_n & i = n \\ e_0 + e_n & i = 0 \end{cases}$$

$$(4) \nu(e_i^*) = \begin{cases} e_i^* & i \neq n \\ e_0^* - e_n^* & i = n \end{cases}$$

(5) lattices of  $F$ -rational characters is spanned by  $\{e_i : 0 < i < n\} \cup \{2e_0 + e_n\}$ .

(6) the lattice of  $F$ -rational cocharacters is spanned by  $\{e_i : 0 \leq i < n\}$ .

*Proof.* One easily checks that the formulae given in (4) define a nontrivial element  $\nu$  of  $S$ , and that the effect on the dual lattice and the roots in it is as described in (2), (3). If  $n \neq 4$ , then the Dynkin diagram has only two automorphisms, and (1) follows. When  $n = 4$ , one must check that  $S$  can not contain any element of order 3. This follows from the fact that such an element would induce an automorphism of  $SO_8(\mathbb{C})$  of order three, and no such automorphism exists. This completes the proof of (1) in this case. Items (5) and (6) are easily checked, given that an element of  $X$  or  $X^\vee$  is  $F$ -rational if and only if it is  $\text{Gal}(\bar{F}/F)$ -stable.  $\square$

By class field theory homomorphisms from  $\text{Gal}(\bar{F}/F)$  to a group with two elements are in one-to-one correspondence with quadratic characters  $\chi : \mathbb{A}_F^\times / F^\times \rightarrow \{\pm 1\}$ . We denote the  $F$ -group corresponding to the character  $\chi$  by  $GSpin_{2n}^\chi$ . The  $F$ -group corresponding to the trivial character is the unique split  $F$ -group having the specified root datum, and is also denoted simply by  $GSpin_{2n}$ .

To save space, the group  $GSpin_m$  will usually be denoted  $G_m$ .

Observe that in either the odd or even case  $e_0^*$  is a generator for the lattice of cocharacters of the center of  $G_m$ .

Because we define  $G_m$  in the manner we do, it comes equipped with a choice of Borel subgroup and maximal torus, as do various reductive subgroups we shall consider below. In each case, we denote the Borel subgroup of the reductive group  $G$  by  $B(G)$ , and the maximal torus by  $T(G)$ .

A straightforward adaptation of the proof of Theorem 16.3.2 of [Spr] shows that there exist surjections  $G_m \rightarrow SO_m$  defined over  $F$ . We fix one such and denote it  $\text{pr}$ . We require that  $\text{pr}$  is such that  $\text{pr}(B(G_m))$  consists of upper triangular matrices.

**Remark 4.3.1.** *For those familiar with the construction of spin groups as a subgroups of the multiplicative groups of Clifford algebras, we remark that our  $GSpin$  groups can be constructed in the same way by including the nonzero scalars in the Clifford algebra as well. However, this description will not play a role for us.*

**4.4. Unipotent subgroups of  $GSpin_m$  and their characters.** The kernel of  $\text{pr}$  consists of semisimple elements. In particular, the number of unipotent elements of a fiber is zero or one, and it's one if and only if the element of  $SO_m$  is unipotent. In other words,  $\text{pr}$  yields a bijection of unipotent elements (indeed, an isomorphism of unipotent subvarieties), and we may specify unipotent elements or subgroups by their images under  $\text{pr}$ . This also defines coordinates for any unipotent element or subgroup, which we use when defining characters. Thus, we write  $u_{ij}$  for the  $i, j$  entry of  $\text{pr}(u)$ .

Above we fixed a specific isomorphism of a subgroup of  $G_{2m}$  with  $GL_m$ . If  $u$  is a unipotent element of this subgroup this identification with an  $m \times m$  matrix gives a second definition of  $u_{ij}$ . This is not a problem, however, as the two definitions agree.

Most of the unipotent groups we consider are subgroups of the maximal unipotent of  $G_m$  consisting of elements  $u$  with  $\text{pr}(u)$  upper triangular. We denote this group  $U_{\max}$ . A complete set of coordinates is  $\{u_{ij} : 1 \leq i < j \leq m - i\}$ . We denote the opposite maximal unipotent by  $\overline{U}_{\max}$ . It consists of all unipotent elements of  $G_m$  such that  $\text{pr}(u)$  is lower triangular.

We fix once and for all a character  $\psi_0$  of  $\mathbb{A}/F$ . We use this character together with the coordinates just above to specify characters of our unipotent subgroups.

When specifying subgroups of  $U_{\max}$  and their characters, the restriction to  $\{(i, j) : 1 \leq i < j \leq m - i\}$  is implicit.

It will also sometimes be necessary to describe unipotent subgroups such that only a few of the entries in the corresponding elements of  $SO_m$  are nonzero. For this purpose we introduce the notation  $e'_{ij} = e_{ij} - e_{m+1-j, m+1-i}$ . One may check that for all  $i \neq j$  and  $a \in F$ , the matrix  $I + ae'_{ij}$  is an element of  $SO_m(F)$ , unless  $m$  is odd and  $i + j = m$ , in which case  $I + ae'_{ij} + \frac{a^2}{2}e_{i, m+1-i} \in SO_m(F)$ .

## 5. "UNIPOTENT PERIODS"

We now introduce the framework within which, we believe, certain of the computations involved in the descent construction can be most easily understood.

Let  $G$  be a reductive algebraic group defined over a number field  $F$ . If  $U$  is a unipotent subgroup of  $G$  and  $\psi_U$  is a character of  $U(F \backslash \mathbb{A})$ , (by which we mean a character of  $U(\mathbb{A})$  trivial on  $U(F)$ ) we define the *unipotent period*  $(U, \psi_U)$  associated to this pair to be given by the formula

$$\varphi^{(U, \psi_U)}(g) := \int_{U(F \backslash \mathbb{A})} \varphi(ug) \psi_U(u) du.$$

Clearly,  $\varphi$  must be restricted to a space of left  $U(F)$ -invariant locally integrable functions.

Let  $\mathcal{U}$  denote the set of unipotent periods. For  $V$  a space of functions defined on  $G(\mathbb{A})$ , put

$$\mathcal{U}^\perp(V) = \{(U, \psi) \in \mathcal{U} : \varphi^{(U, \psi)} \equiv 0 \ \forall \varphi \in V\}.$$



When  $V$  is the space of a representation  $\pi$  we will employ also the notation  $\mathcal{U}^\perp(\pi)$ . We also employ the notation  $(U, \psi) \perp V$  for  $(U, \psi) \in \mathcal{U}^\perp(V)$  and similarly  $(U, \psi) \perp \pi$ .

We also require a vocabulary to express relationships among unipotent periods. We shall say that

$$(U, \psi_U) \in \langle (U_1, \psi_{U_1}), \dots, (U_n, \psi_{U_n}), \dots \rangle$$

if  $V \perp (U_i, \psi_{U_i}) \forall i \Rightarrow V \perp (U, \psi_U)$ . Clearly, if  $(U_1, \psi_{U_1}) \in \langle (U_2, \psi_2), (U_3, \psi_3) \rangle$ , and  $(U_2, \psi_2) \in \langle (U_4, \psi_4), (U_5, \psi_5) \rangle$  then  $(U_1, \psi_1) \in \langle (U_3, \psi_3), (U_4, \psi_4), (U_5, \psi_5) \rangle$ .

We also use notation  $(U_1, \psi_1)|(U_2, \psi_2)$ , or the language “ $(U_1, \psi_1)$  divides  $(U_2, \psi_2)$ ,” “ $(U_2, \psi_2)$  is divisible by  $(U_1, \psi_1)$ ” for  $(U_2, \psi_2) \in \langle (U_1, \psi_1) \rangle$ . Finally,  $(U_1, \psi_1) \sim (U_2, \psi_2)$  means  $(U_1, \psi_1)|(U_2, \psi_2)$  and  $(U_2, \psi_2)|(U_1, \psi_1)$ . This is an equivalence relation and we shall refer to unipotent periods which are related in this way as “equivalent.”

It is *sometimes* possible to compose unipotent periods. Specifically, if  $f^{(U_1, \psi_1)}$  is left-invariant by  $U_2(F)$ , then one may consider  $(f^{(U_1, \psi_1)})^{(U_2, \psi_2)}$ . We denote the composite by  $(U_2, \psi_2) \circ (U_1, \psi_1)$ .

If  $(U, U)$  is the commutator subgroup of  $U$ , then  $U/(U, U)$  is a vector group in the sense of [Spr], p. 51. The normalizer  $N_G(U)$  of  $U$  in  $G$  acts on  $U/(U, U)$ . This is an  $F$ -rational representation of  $N_G(U)$ . Denote the dual  $F$ -rational representation of  $N_G(U)$  by  $(U/(U, U))^*$ . Then the choice of  $\psi_0$  identifies  $F$  with the space of characters of  $F \backslash \mathbb{A}$  and thus identifies  $(U/(U, U))^*(F)$  with the space of characters of  $U(\mathbb{A})$  which are trivial on  $U(F)$ . This gives this space the additional structure of an  $F$ -rational representation of  $N_G(U)$ .

Now, suppose that  $U$  is the unipotent radical of a parabolic  $P$  of  $G$  with Levi  $M$ .

**Lemma 5.0.1.** *Then  $(U/(U, U))^*$  is isomorphic, as an  $F$ -rational representation of  $M$ , to  $\overline{U}/(\overline{U}, \overline{U})$ , where  $\overline{U}$  denotes the unipotent radical of the parabolic  $\overline{P}$  of  $G$  opposite to  $P$ .*

*Proof.* The exponential map induces an  $M$ -equivariant isomorphism  $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}] \rightarrow U/(U, U)$ , defined over  $F$ . The Killing form  $B$  induces  $M$ -equivariant isomorphisms  $\mathfrak{u}^* \rightarrow \overline{\mathfrak{u}}$  and  $([\mathfrak{u}, \mathfrak{u}])^* \rightarrow [\overline{\mathfrak{u}}, \overline{\mathfrak{u}}]$ . If  $\mathfrak{v} := \{X \in \overline{\mathfrak{u}} : B(X, Y) = 0 \forall Y \in [\mathfrak{u}, \mathfrak{u}]\}$ , then  $\mathfrak{v}$  is canonically identified with  $(\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}])^*$ , and maps isomorphically onto  $\overline{U}/(\overline{U}, \overline{U})$ .  $\square$

This idea can be extended to more general unipotent groups. See section 5.2.1.

For  $\vartheta$  a character of  $U(\mathbb{A})$  which is trivial on  $U(F)$ , let  $M^\vartheta$  denote the stabilizer of  $\vartheta$  (regarded as a point in  $\overline{U}/(\overline{U}, \overline{U})(F)$ ) in  $M$ . So  $M^\vartheta$  is an algebraic subgroup of  $M$  defined over  $F$ .

**Definition 5.0.2.** *In the set up of the previous paragraph, let  $FC^\vartheta : C^\infty(G(F \backslash \mathbb{A})) \rightarrow C^\infty(M^\vartheta(F \backslash \mathbb{A}))$  be given by*

$$FC^\vartheta(\varphi)(m) = \varphi^{(U, \vartheta)}(m) = \int_{U(F \backslash \mathbb{A})} \varphi(um) \vartheta(u) du.$$

Clearly,

**Lemma 5.1.** *The map  $FC^\vartheta$  is an  $M^\vartheta(\mathbb{A})$ -equivariant map.*

**5.1. A Lemma Regarding Unipotent Periods.** There is a natural action of  $G(F)$  on the space of unipotent periods  $\mathcal{U}$  given by  $\gamma \cdot (U, \psi) = (\gamma U \gamma^{-1}, \gamma \cdot \psi)$  where  $\gamma \cdot \psi(u) = \psi(\gamma^{-1} u \gamma)$ . We shall refer to this action as “conjugation.” Obviously, unipotent periods which are conjugate are equivalent.

**Lemma 5.1.1.** *Suppose  $U_1 \supset U_2 \supset (U_1, U_1)$  are unipotent subgroups of a reductive algebraic group  $G$ . Suppose  $H$  is a subgroup of  $G$  and let  $f$  be a smooth left  $H(F)$ -invariant function on  $G(\mathbb{A})$ . Suppose  $\psi_2$  is a character of  $U_2$  such that  $\psi_2|_{(U_1, U_1)} \equiv 0$ . Then the set  $\text{res}^{-1}(\psi_2)$  of characters of  $U_1$  such that the restriction to  $U_2$  is  $\psi_2$  is nontrivial. (Here “res” is for “restriction” not “residue.”) The elements of  $\text{res}^{-1}(\psi_2)$  are permuted by the action of  $N_H(U_1)(F)$ . The following are equivalent.*

- (1)  $f^{(U_2, \psi_2)} \equiv 0$
- (2)  $f^{(U_1, \psi_1)} \equiv 0 \forall \psi_1 \in \text{res}^{-1}(\psi_2)$



(3) For each  $N_H(U_1)(F)$ -orbit  $\mathcal{O}$  in  $\text{res}^{-1}(\psi_2)$   $\exists \psi_1 \in \mathcal{O}$  with  $f^{(U_1, \psi_1)} \equiv 0$

*Proof.* It is obvious that 1 implies 2 and 3, and that 2 and 3 are equivalent. Consider

$$f^{(U_2, \psi_2)}(u_1 g) = \int_{U_2(F \backslash \mathbb{A})} f(u_2 u_1 g) \psi_2(u_2) du_2,$$

regarded as a function of  $u_1$ . It is left  $u_2$  invariant and hence gives rise to a function of the compact abelian group  $U_2(\mathbb{A})U_1(F) \backslash U_1(\mathbb{A})$ . Denote this function by  $\phi(u_1)$ . Then

$$\phi(0) = \sum_{\chi} \int_{U_2(\mathbb{A})U_1(F) \backslash U_1(\mathbb{A})} \phi(u_1) \chi(u_1) du_1,$$

where “0” denotes the identity in  $U_2(\mathbb{A})U_1(F) \backslash U_1(\mathbb{A})$ , and the sum is over characters of  $U_2(\mathbb{A})U_1(F) \backslash U_1(\mathbb{A})$ . This, in turn, is equal to

$$\sum_{\chi} \int_D \int_{U_2(F \backslash \mathbb{A})} f(u_2 u_1 g) \psi_2(u_2) du_2 \chi(u_1) du_1,$$

for  $D$  a fundamental domain for the above quotient in  $U_1(\mathbb{A})$ . The group  $U_1/(U_1, U_1)(F)$  is an  $F$ -vector space (cf. section 5) which can be decomposed into  $U_2/(U_1, U_1)(F)$  and a complement. The  $F$ -dual of this vector space is identified, via the choice of  $\psi_0$ , with the space of characters of  $U_1(\mathbb{A})$  which are trivial on  $U_1(F)$ . It follows that the sum above is equal to

$$= \sum_{\psi_1 \in \text{res}^{-1}(\psi_2)} \int_{U_1(F \backslash \mathbb{A})} f(u_1 g) \psi_1(u_1) du_1.$$

The matter of replacing the sum over  $\chi$  by one over  $\psi_1 \in \text{res}^{-1}(\psi_2)$  is clear from regarding  $U_1/(U_1, U_1)(F)$  as a vector space which can be decomposed into  $U_2/(U_1, U_1)$  and a complement. Now  $2 \Rightarrow 1$  is immediate.  $\square$

**Corollary 5.1.2.** *If  $N_G(H)$  permutes the elements of  $\text{res}^{-1}(\psi_2)$  transitively, then  $(U_2, \psi_2) \sim (U_1, \psi_1)$  for every  $\psi_1 \in \text{res}^{-1}(\psi_2)$ .*

**Definition 5.1.3.** *Many of the applications of the above corollary are of a special type, and it will be convenient to introduce a term for them. The special situation is the following: one has three unipotent periods  $(U_i, \psi_i)$  for  $i = 1, 2, 3$ , such that  $U_2 = U_1 \cap U_3$  and  $\psi_1|_{U_2} = \psi_3|_{U_2} = \psi_2$ . Furthermore,  $U_1$  normalizes  $U_3$  and permutes transitively the set of characters  $\psi'_3$  such that  $\psi'_3|_{U_2} = \psi_2$ , and the same is true with the roles of 1 and 3 reversed. In this situation, the identity*

$$(U_1, \psi_1) \sim (U_2, \psi_2) \sim (U_3, \psi_3),$$

*(which follows from Corollary 5.1.2) will be called a **swap**, and we say that  $(U_1, \psi_1)$  “may be swapped for”  $(U_3, \psi_3)$ , and vice versa.*

**Remark 5.1.4.** *This notion of a “swap” is essentially an alternate take on “exchange of roots,” as described in [GL], section 2.1 (and references therein; see also [GRS7]).*

## 5.2. Relation of unipotent periods via theta functions.

**5.2.1. Initial remarks.** We described above how a character of  $U(F \backslash \mathbb{A})$ , may be thought of as an element of an  $F$ -vector space equipped with an algebraic action of  $N_G(U)$ . It was further shown how to identify such characters with the  $F$  points of a distinguished subspace  $\mathfrak{v}$  of  $\mathfrak{g}$  in the special case when  $U$  is the unipotent radical of a parabolic subgroup. This is advantageous, because the space  $\mathfrak{g}_F$  is equipped with an action of  $G(F)$ , which is compatible with the action of  $G(F)$  on unipotent periods by conjugation (as in 5.1).

The same basic idea may be extended to arbitrary unipotent subgroups using a suitable involution, as we now explain. First, every unipotent subgroup of  $G$  is conjugate to a subgroup of  $U_{\max}$ .

So, assume  $U \subset U_{\max}$ . There is an automorphism  $\iota$  of  $G$ , unique up to conjugation by an element of  $T$ , which maps  $T$  to  $T$  in such a fashion that  $\iota(t)^\alpha = t^{-\alpha}$  for all  $t \in T$  and all roots  $\alpha$ , and which maps the one-dimensional unipotent subgroup  $U_\alpha$  to  $U_{-\alpha}$ . For  $GSpin$  groups we can normalize  $\iota$  by requiring that the automorphism induced on  $SO_m$  be transpose inverse. The automorphism  $\iota$  induces an automorphism of  $\mathfrak{g}$  which we denote by the same letter. Define  $\mathbb{B}'(X, Y) = \mathbb{B}(X, \iota(Y))$  where  $\mathbb{B}$  is the Killing form. Then the restriction of  $\mathbb{B}'$  to  $\mathfrak{u}_{\max}$  is nondegenerate. If  $U$  is any unipotent subgroup of  $G$ , define  $\overline{U} = \iota(U)$ . Note that if  $U$  is the unipotent radical of a parabolic subgroup, then  $\overline{U}$  is the unipotent radical of the opposite parabolic, so this definition of  $\overline{U}$  extends the previous usage. Then  $\mathbb{B}$  identifies the perp space of  $[\mathfrak{u}, \mathfrak{u}]$  in  $\overline{\mathfrak{u}}$  with  $(U/(U, U))^*$ . Thus, each character of  $U(F) \backslash U(\mathbb{A})$  is attached to an element of  $\overline{\mathfrak{u}}_F \subset \mathfrak{g}_F$ .

Let  $(U_1, \psi_1)$  and  $(U_2, \psi_2)$  be two unipotent periods and suppose that the equivalence  $(U_1, \psi_1) \sim (U_2, \psi_2)$  may be proved using conjugation and swapping. Then it's fairly easy to see that  $\psi_1$  and  $\psi_2$  correspond to points in the same orbit of  $G(F)$  acting on  $\mathfrak{g}_F$ . The manner in which  $U_1$  and  $U_2$  are related is not as easy to describe, but one may note for example that they will have the same dimension.

The purpose of this section is to explain how the theory of the Weil representation may be used to obtain subtler relations among unipotent periods. This method was shown to us by David Ginzburg.

**5.2.2. Preliminaries on the Jacobi group and Weil representation.** Let  $\mathcal{H}$  denote the Heisenberg group in three variables, which we define as the set of triples  $(x, y, z) \in \mathbb{G}_a^3$  equipped with the product  $(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2 - y_1 x_2)$ . The group is two-step nilpotent: its center,  $Z$  is equal to its commutator subgroup, and consists of all elements of the form  $(0, 0, z)$ . The quotient  $\mathcal{H}/Z$  is a vector group. The commutator defines a skew-symmetric bilinear form  $\mathcal{H}/Z \times \mathcal{H}/Z \rightarrow Z$ .

The group  $SL_2$  may be identified with the group of all automorphisms of  $\mathcal{H}/Z$  which fix this symplectic form, or, equivalently, with the group of all automorphisms of  $\mathcal{H}$  which restrict to the identity on  $Z$ . The semidirect product  $\mathcal{H} \rtimes SL_2$  of  $\mathcal{H}$  and  $SL_2$  is sometimes called the Jacobi group. Following [BS], we denote this group by  $G^J$  and identify it with  $\left\{ \begin{pmatrix} 1 & * & * \\ & g_0 & * \\ & & 1 \end{pmatrix} : g_0 \in SL_2 \right\} \subset Sp_4$ .

The group  $\mathcal{H}(\mathbb{A})$  has a unique isomorphism class of unitary representations such that the central character is  $(0, 0, z) \mapsto \psi_0(z)$ . Any representation in this class extends uniquely to a projective representation of  $G^J(\mathbb{A})$ , or to a genuine representation of  $\widetilde{G}^J(\mathbb{A}) := \mathcal{H}(\mathbb{A}) \rtimes \widetilde{SL}_2(\mathbb{A})$ . Here  $\widetilde{SL}_2(\mathbb{A})$  denotes the metaplectic double cover of  $SL_2(\mathbb{A})$ .

There is a representation  $\omega_{\psi_0}$  known as the Weil representation which is in this class, given by action on the space  $\mathcal{S}(\mathbb{A})$  of Schwartz functions on  $\mathbb{A}$  such that

$$(5.2.1) \quad \omega_{\psi_0}(0, y, 0)\phi(\xi) = \psi_0(y\xi)\phi(\xi), \quad \omega_{\psi_0}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\phi(\xi) = \psi_0(x\xi^2)\phi(x) \text{ and}$$

$$\omega_{\psi_0}(x, 0, 0)\phi(x) = \phi(\xi + x), \quad (x, y, \xi \in \mathbb{A}, \phi \in \mathcal{S}(\mathbb{A})).$$

(We identify  $\mathcal{H}(\mathbb{A})$  and  $\widetilde{SL}_2(\mathbb{A})$  with their images in  $\mathcal{H}(\mathbb{A}) \rtimes \widetilde{SL}_2(\mathbb{A})$ , and write  $\omega_{\psi_0}(x, y, z)$  or  $\omega_{\psi_0}(g)$  as appropriate. The group  $\left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in \mathbb{A} \right\}$  lifts into  $\widetilde{SL}_2(\mathbb{A})$  and we identify it with its image, writing  $\omega_{\psi_0}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$  as well. )

It is known that  $SL_2(F)$  lifts into  $\widetilde{SL}_2(\mathbb{A})$ , and hence  $G^J(F)$  lifts into  $\widetilde{G}^J(\mathbb{A})$ . The representation  $\omega_{\psi_0}$  has an automorphic realization given by theta functions

$$\theta_\phi^{\psi_0}(g) = \sum_{\xi \in F} \omega_{\psi_0}(g)\phi(\xi), \quad (g \in \widetilde{G}^J(\mathbb{A})).$$

For each  $\phi \in \mathcal{S}(\mathbb{A})$ , the corresponding theta function is a genuine function  $G^J(F) \backslash \tilde{G}^J(\mathbb{A})$ , i.e., it does not factor through the projection to  $G^J(\mathbb{A})$ .

It is further known that  $\{(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}) : x \in \mathbb{A}\}$  lifts into  $\widetilde{SL}_2(\mathbb{A})$ . A smooth function  $f : SL_2(F) \backslash \widetilde{SL}_2(\mathbb{A}) \rightarrow \mathbb{C}$  or  $G^J(F) \backslash \tilde{G}^J(\mathbb{A}) \rightarrow \mathbb{C}$  has a Fourier expansion

$$f = \sum_{a \in F} f^{\psi_a}, \quad \text{where} \quad f^{\psi_a}(\tilde{h}) := \int_{(F \backslash \mathbb{A})} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tilde{h}\right) \psi(ax) dx.$$

Since  $U(\mathbb{A})$  and  $SL_2(F)$  generate  $\widetilde{SL}_2(\mathbb{A})$ , it follows that

$$f^{\psi_a} = 0 \quad \forall a \in F^\times \implies f \text{ is constant on } \widetilde{SL}_2(\mathbb{A})$$

and thus

$$(5.2.2) \quad f \text{ is genuine} \implies f^{\psi_a} \neq 0 \text{ for some } a \in F^\times.$$

5.2.3. *Fourier-Jacobi coefficients.* Define the Fourier-Jacobi coefficient mapping

$$FJ^{\psi_0} : C^\infty(G^J(F) \backslash \tilde{G}^J(\mathbb{A})) \times \mathcal{S}(\mathbb{A}) \rightarrow C^\infty(SL_2(F) \backslash \widetilde{SL}_2(\mathbb{A})).$$

$$FJ^{\psi_0}(f, \phi)(\tilde{g}_0) := \int_{\mathcal{H}(F \backslash \mathbb{A})} f(u\tilde{g}_0) \theta_\phi^{\psi_0}(u\tilde{g}_0) du, \quad \left(\tilde{g}_0 \in \widetilde{SL}_2(\mathbb{A}), f \in C^\infty(G^J(F) \backslash \tilde{G}^J(\mathbb{A})), \phi \in \mathcal{S}(\mathbb{A})\right)$$

Recall that a function  $G^J(F) \backslash \tilde{G}^J(\mathbb{A}) \rightarrow \mathbb{C}$  (resp.  $SL_2(F) \backslash \widetilde{SL}_2(\mathbb{A}) \rightarrow \mathbb{C}$ ) is said to be genuine if it does not factor through the projection to  $G^J(F \backslash \mathbb{A})$  (resp.  $SL_2(F \backslash \mathbb{A})$ ). Since  $\theta_\phi^{\psi_0}$  is genuine, it follows that  $FJ^{\psi_0}(f, \phi)$  is genuine if and only if  $f$  is not, with the only exception being that  $FJ^{\psi_0}(f, \phi)$  may equal zero (which is of course not genuine) when  $f$  is not genuine.

Now, let

$$U_{\text{Si}} = \left\{ \begin{pmatrix} I & X \\ & I \end{pmatrix} : X = \begin{pmatrix} y & z \\ r & y \end{pmatrix} \right\}, \quad \psi_{U_{\text{Si}}, a} := \psi_0(z + ar).$$

and for  $f \in C^\infty(G^J(F) \backslash \tilde{G}^J(\mathbb{A}))$ , define

$$f^{(U_{\text{Si}}, \psi_{U_{\text{Si}}, a})}(\tilde{g}_0) := \int_{U_{\text{Si}}(F \backslash \mathbb{A})} f(u\tilde{g}_0) \psi_{U_{\text{Si}}}(u) du$$

**Lemma 5.2.3.** *Given  $f \in C^\infty(G^J(F) \backslash \tilde{G}^J(\mathbb{A}))$ , we have*

$$\left(FJ^{\psi_0}(\phi, f)\right)^{\psi_a}(\tilde{g}_0) = \int_{\mathbb{A}} f^{(U_{\text{Si}}, \psi_{U_{\text{Si}}, a})}((x, 0, 0)\tilde{g}_0) [\omega_{\psi_0}(\tilde{g}_0)\phi](x) dx$$

*Proof.*

$$\begin{aligned} \left(FJ^{\psi_0}(\phi, f)\right)^{\psi_a}(\tilde{g}_0) &= \int_{F \backslash \mathbb{A}} FJ^{\psi_0}(\phi, f) \left( \begin{pmatrix} 1 & r_1 \\ & 1 \end{pmatrix} \tilde{g}_0 \right) \psi_0(ar_1) dr_1 \\ &= \int_{F \backslash \mathbb{A}} \int_{\mathcal{H}(F \backslash \mathbb{A})} f \left( u \begin{pmatrix} 1 & r_1 \\ & 1 \end{pmatrix} \tilde{g}_0 \right) \theta_\phi^{\psi_0} \left( u \begin{pmatrix} 1 & r_1 \\ & 1 \end{pmatrix} \tilde{g}_0 \right) du \psi_0(ar_1) dr_1. \end{aligned}$$

It is convenient to identify unipotent elements with their images in  $Sp_4$ . In this notation we have

$$(5.2.4) \quad \int_{F \backslash \mathbb{A}} \int_{F \backslash \mathbb{A}} \int_{F \backslash \mathbb{A}} \int_{F \backslash \mathbb{A}} f \left( \begin{pmatrix} 1 & x & y + xr_1 & z \\ & 1 & r_1 & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \tilde{g}_0 \right) \theta_\phi^{\psi_0} \left( \begin{pmatrix} 1 & x & y + xr_1 & z \\ & 1 & r_1 & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \tilde{g}_0 \right) \psi_0(ar_1) dx dy dz dr_1$$

Now, for any  $g \in \tilde{G}^J(\mathbb{A})$ ,

$$\theta_\phi^{\psi_0}(g) = \sum_{\xi \in F} \omega_{\psi_0}(g) \phi(\xi) = \omega_{\psi_0} \left( \begin{pmatrix} 1 & \xi & & \\ & 1 & & \\ & & 1 - \xi & \\ & & & 1 \end{pmatrix} g \right) \phi(0).$$

Plug this in, use the invariance of  $f$  by  $G^J(F)$ , and collapse summation in  $\xi$  with integration in  $x$ . It follows that (5.2.4) is equal to

$$\int_{\mathbb{A}} \int_{F \backslash \mathbb{A}} \int_{F \backslash \mathbb{A}} \int_{F \backslash \mathbb{A}} f \left( \begin{pmatrix} 1 & x & y + xr_1 & z \\ & 1 & r_1 & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \tilde{g}_0 \right) \theta_\phi^{\psi_0} \left( \begin{pmatrix} 1 & x & y + xr_1 & z \\ & 1 & r_1 & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \tilde{g}_0 \right) \psi_0(ar_1) dy dz dr_1 dx$$

One has

$$\begin{pmatrix} 1 & x & y + xr_1 & z \\ & 1 & r_1 & y \\ & & 1 & -x \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & y' & z' \\ & 1 & r_1 & y' \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix}, y' = y + xr_1, z' = z + xy'$$

and from (5.2.1)

$$\omega_{\psi_0} \begin{pmatrix} 1 & & y' & z' \\ & 1 & r_1 & y' \\ & & 1 & \\ & & & 1 \end{pmatrix} \phi_1(0) = \psi(z') \phi_1(0) \quad (\forall y', z', r_1 \in \mathbb{A}, \phi_1 \in \mathcal{S}(\mathbb{A})).$$

The result follows.  $\square$

**Corollary 5.2.5.** *For  $f \in C^\infty(G^J(F) \backslash \tilde{G}^J(\mathbb{A}))$ , we have*

$$f^{(U_{\text{Si}}, \psi_{U_{\text{Si}}, a})} \equiv 0 \iff FJ^{\psi_0}(\phi, f)^{\psi_a} \equiv 0 \quad \forall \phi \in \mathcal{S}(\mathbb{A}).$$

*Proof.* This follows from lemma 5.2.3, because a smooth function whose integral against every Schwartz function is zero is the zero function (and vice versa).  $\square$

**Corollary 5.2.6.** *For  $f \in C^\infty(G^J(F) \backslash \tilde{G}^J(\mathbb{A}))$ , not genuine, we have*

$$f^{(U_{\text{Si}}, \psi_{U_{\text{Si}}, 0})} \not\equiv 0 \implies f^{(U_{\text{Si}}, \psi_{U_{\text{Si}}, a})} \not\equiv 0, \text{ some } a \in F^\times.$$

*Proof.* Indeed, for each  $\phi \in \mathcal{S}(\mathbb{A})$ , the function  $FJ^{\psi_0}(\phi, f)$  is either 0 or genuine. If  $f^{(U_{\text{Si}}, \psi_{U_{\text{Si}}, 0})} \not\equiv 0$  then it follows from corollary 5.2.5 that  $FJ^{\psi_0}(\phi, f)$  is nonzero for some  $\phi$ . It then follows from (5.2.2) that  $FJ^{\psi_0}(\phi, f)^{\psi_a}$  is nonzero for some  $a \in F^\times$ , and corollary 5.2.5 completes the proof.  $\square$

## Part 2. Odd case

### 6. NOTATION AND STATEMENT

We are now ready to formulate the main result of the paper in the odd case.

**Theorem (MAIN THEOREM: ODD CASE).** *For  $r \in \mathbb{N}$ , take  $\tau_1, \dots, \tau_r$  to be irreducible unitary automorphic cuspidal representations of  $GL_{2n_1}(\mathbb{A}), \dots, GL_{2n_r}(\mathbb{A})$ , respectively, and let  $\tau = \tau_1 \boxplus \dots \boxplus \tau_r$ . Let  $\omega$  denote a Hecke character. Suppose that*

- $\tau_i$  is  $\bar{\omega}$ -symplectic for each  $i$ , and
- $\tau_i \cong \tau_j \implies i = j$ .

Then there exists an irreducible generic cuspidal automorphic representation  $\sigma$  of  $GSpin_{2n+1}(\mathbb{A})$  such that

- $\sigma$  weakly lifts to  $\tau$ , and
- the central character of  $\sigma$  is  $\omega$ .

In fact, a refinement of this theorem with an explicit description of  $\sigma$  is given in theorem 9.2.1, and proved in section 9.

**Remark 6.0.7.** *The case  $n = 1$  is trivial because  $GSpin_3 = GSp_2 = GL_2$ , so the inclusion  $r$  is simply the identity map. Clearly,  $r$  must be one and  $\sigma = \tau_1$ . Henceforth, we assume  $n \geq 2$ . The careful reader will find places where this assumption is crucial to the validity of the argument.*

**6.1. Siegel Parabolic.** We will construct an Eisenstein series on  $G_{2m}$  induced from a standard parabolic  $P = MU$  such that  $M$  is isomorphic to  $GL_m \times GL_1$ . There are two such parabolics. We choose the one in which we delete the root  $e_{m-1} + e_m$  and the coroot  $e_{m-1}^* + e_m^* - e_0^*$  from the based root datum. We shall refer to this parabolic as the “Siegel.”

- Remarks 6.1.1.**
- We can identify the based root datum of the Levi  $M$  with that of  $GL_m \times GL_1$  in such a fashion that  $e_0$  corresponds to  $GL_1$  and does not appear at all in  $GL_m$ . We can then identify  $M$  itself with  $GL_m \times GL_1$  via a particular choice of isomorphism which is compatible with this and with the usual usage of  $e_i, e_i^*$  for characters, cocharacters of the standard torus of  $GL_m$ .
  - Having made this identification, a Levi  $M'$  which is contained in  $M$  will be identified with  $GL_1 \times GL_{m_1} \times \dots \times GL_{m_k}$ , (for some  $m_1, \dots, m_k \in \mathbb{N}$  that add up to  $m$ ) in the natural way:  $GL_1$  is identified with the  $GL_1$  factor of  $M$ , and then  $GL_{m_1} \times \dots \times GL_{m_k}$  is identified with the subgroup of  $M$  corresponding to block diagonal elements with the specified block sizes, in the specified order.
  - The lattice of rational characters of  $M$  is spanned by the maps  $(g, \alpha) \mapsto \alpha$  and  $(g, \alpha) \mapsto \det g$ . Restriction defines an embedding  $X(M) \rightarrow X(T)$ , which sends these maps to  $e_0$  and  $(e_1 + \dots + e_m)$ , respectively. By abuse of notation, we shall refer to the rational character of  $M$  corresponding to  $e_0$  as  $e_0$  as well.
  - $\delta_P(g, \alpha) = \det g^{(m-1)}$ , with  $\delta_P$  the modulus function of  $P$ .

The group  $G_{2n}$  has an involution  $\dagger$  which reverses the last two simple roots. The effect is such that

$$\mathrm{pr}(\dagger g) = \begin{pmatrix} I_{n-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & I_{n-1} \end{pmatrix} \mathrm{pr}(g) \begin{pmatrix} I_{n-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & I_{n-1} \end{pmatrix}.$$

As is well known, there is a group  $Pin_{4n} \supset Spin_{4n}$  such that  $\mathrm{pr}$  extends to a two-fold covering  $Pin_{4n} \rightarrow O_{4n}$ . The involution  $\dagger$  can be realized as conjugation by a preimage of the above permutation matrix.

## 6.2. Weyl group of $GSpin_{2m}$ ; it's action on standard Levis and their representations.

**Lemma 6.2.1.** *The Weyl group of  $G_m$  is canonically identified with that of  $SO_m$ .*

*Proof.* For this lemma only, let  $T$  denote the torus of  $SO_m$  and  $\tilde{T}$  that of  $G_m$ . Then the following diagram commutes:

$$\begin{array}{ccc} Z_{G_m}(\tilde{T}) & \longrightarrow & N_{G_m}(\tilde{T}) \\ \downarrow & & \downarrow \\ Z_{SO_m}(T) & \longrightarrow & N_{SO_m}(T). \end{array}$$

Both horizontal arrows are inclusions and both vertical arrows are pr.  $\square$

One easily checks that every element of the Weyl group of  $SO_{2n}$  is represented by a permutation matrix. We denote the permutation associated to  $w$  also by  $w$ . The set of permutations  $w$  obtained is precisely the set of permutations  $w \in \mathfrak{S}_{2n}$  satisfying,

- (1)  $w(2n+1-i) = 2n+1-w(i)$  and
- (2)  $\det w = 1$  when  $w$  is written as a  $2n \times 2n$  permutation matrix.

It is well known that the Weyl group of  $SO_{2n}$  (or  $G_{2n}$ ) is isomorphic to  $\mathfrak{S}_n \rtimes \{\pm 1\}^{n-1}$ . To fix a concrete isomorphism, we identify  $p \in \mathfrak{S}_n$  with an  $n \times n$  matrix in the usual way, and then with

$$\begin{pmatrix} p & \\ & {}^t p^{-1} \end{pmatrix} \in SO_{2n}.$$

We identify  $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_{n-1}) \in \{\pm 1\}^{n-1}$  with the permutation  $p$  of  $\{1, \dots, 2n\}$  such that

$$p(i) = \begin{cases} i & \text{if } \epsilon_i = 1 \\ 2n+1-i & \text{if } \epsilon_i = -1, \end{cases}$$

where  $\epsilon_n$  is defined to be  $\prod_{i=1}^{n-1} \epsilon_i$ . We then identify  $(p, \underline{\epsilon}) \in \mathfrak{S}_n \times \{\pm 1\}^{n-1}$  (direct product of sets) with  $p \cdot \underline{\epsilon} \in W_{SO_{2n}}$ .

With this identification made,

(6.2.2)

$$(p, \underline{\epsilon}) \cdot \begin{pmatrix} t_1 & & & & \\ & \ddots & & & \\ & & t_n & & \\ & & & t_n^{-1} & \\ & & & & \ddots \\ & & & & & t_1^{-1} \end{pmatrix} \cdot (p, \underline{\epsilon})^{-1} = \begin{pmatrix} t_{p^{-1}(1)}^{\epsilon_{p^{-1}(1)}} & & & & \\ & \ddots & & & \\ & & t_{p^{-1}(n)}^{\epsilon_{p^{-1}(n)}} & & \\ & & & t_{p^{-1}(n)}^{-\epsilon_{p^{-1}(n)}} & \\ & & & & \ddots \\ & & & & & t_{p^{-1}(1)}^{-\epsilon_{p^{-1}(1)}} \end{pmatrix}.$$

**Lemma 6.2.3.** *Let  $(p, \underline{\epsilon}) \in \mathfrak{S}_n \rtimes \{\pm 1\}^{n-1}$  be identified with an element of  $W_{SO_{2m}} = W_{G_{2m}}$  as above. Then the action on the character and cocharacter lattices of  $G_{2m}$  is given as follows:*

$$\begin{aligned} (p, \underline{\epsilon}) \cdot e_i &= \begin{cases} e_{p(i)} & i > 0, \epsilon_{p(i)} = 1, \\ -e_{p(i)} & i > 0, \epsilon_{p(i)} = -1, \\ e_0 + \sum_{\epsilon_{p(i)} = -1} e_{p(i)} & i = 0. \end{cases} \\ (p, \underline{\epsilon}) \cdot e_i^* &= \begin{cases} e_{p(i)}^* & i > 0, \epsilon_{p(i)} = 1, \\ e_0^* - e_{p(i)}^* & i > 0, \epsilon_{p(i)} = -1, \\ e_0^* & i = 0. \end{cases} \end{aligned}$$



**Remark 6.2.4.** *Much of this can be deduced from (6.2.2), keeping in mind that  $w \in W_G$  acts on cocharacters by  $(w \cdot \varphi)(t) = w\varphi(t)w^{-1}$  and on characters by  $(w \cdot \chi)(t) = \chi(w^{-1}tw)$ . However, it is more convenient to give a different proof.*

*Proof.* Let  $\alpha_i = e_i - e_{i+1}$ ,  $i = 1$  to  $n-1$  and  $\alpha_n = e_{n-1} + e_n$ . Let  $s_i$  denote the elementary reflection in  $W_{G_{2n}}$  corresponding to  $\alpha_i$ . Then it is easily verified that  $s_1, \dots, s_{n-1}$  generate a group isomorphic to  $\mathfrak{S}_n$  which acts on  $\{e_1, \dots, e_n\} \in X(T)$  and  $\{e_1^*, \dots, e_n^*\} \in X^\vee(T)$  by permuting the indices and acts trivially on  $e_0$  and  $e_0^*$ . Also

$$\begin{aligned} s_n \cdot e_i &= \begin{cases} e_i & i \neq n-1, n, 0 \\ e_0 + e_n + e_{n-1} & i = 0 \\ -e_n & i = n-1 \\ -e_{n-1} & i = n \end{cases} \\ s_n \cdot e_i^* &= \begin{cases} e_i^* & i \neq n-1, n \\ e_0^* - e_n^* & i = n-1 \\ e_0^* - e_{n-1}^* & i = n. \end{cases} \end{aligned}$$

If  $\underline{\epsilon} \in \{\pm 1\}^{n-1}$  is such that  $\#\{i : \epsilon_i = -1\} = 1$  or  $2$ , then  $\underline{\epsilon}$  is conjugate to  $s_n$  by an element of the subgroup isomorphic to  $\mathfrak{S}_n$  generated by  $s_1, \dots, s_{n-1}$ . An arbitrary element of  $\{\pm 1\}^{n-1}$  is a product of elements of this form, so one is able to deduce the assertion for general  $(p, \underline{\epsilon})$ .  $\square$

Observe that the  $\mathfrak{S}_n$  factor in the semidirect product is precisely the Weyl group of the Siegel Levi.

In the study of Jacquet modules of induced representations as well as in the study of intertwining operators and Eisenstein series (e.g., section 8 below), one encounters a certain subset of the Weyl group associated to a standard Levi,  $M$ . Specifically,

$$W(M) := \left\{ w \in W_{G_{2n}} \mid \begin{array}{l} w \text{ is of minimal length in } w \cdot W_M \\ wMw^{-1} \text{ is a standard Levi of } G_{2n} \end{array} \right\}.$$

For our purposes, it is enough to consider the case when  $M$  is a subgroup of the Siegel Levi. In this case it is isomorphic to  $GL_{m_1} \times \dots \times GL_{m_r} \times GL_1$  for some integers  $m_1, \dots, m_r$  which add up to  $n$ , and we shall only need to consider the case when  $m_i$  is even for each  $i$ . (This, of course, forces  $n$  to be even as well.)

**Lemma 6.2.5.** *For each  $w \in W(M)$  with  $M$  as above, there exist a permutation  $p \in \mathfrak{S}_r$  and an element  $\underline{\epsilon} \in \{\pm 1\}^r$  such that, if  $m \in M = (g, \alpha)$  with  $\alpha \in GL_1$  and*

$$g = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_r \end{pmatrix} \in GL_n,$$

then

$$wmw^{-1} = (g', \alpha \cdot \prod_{\epsilon_i = -1} \det g_i) \quad g' = \begin{pmatrix} g'_1 & & \\ & \ddots & \\ & & g'_r \end{pmatrix},$$

where

$$g'_i \approx \begin{cases} g_{p^{-1}(i)} & \text{if } \epsilon_{p^{-1}(i)} = 1, \\ t g_{p^{-1}(i)}^{-1} & \text{if } \epsilon_{p^{-1}(i)} = -1. \end{cases}$$

Here  $\approx$  has been used to denote equality up to an inner automorphism. The map  $(p, \underline{\epsilon}) \mapsto w$  is a bijection between  $W(M)$  and  $\mathfrak{S}_r \times \{\pm 1\}^r$ . (Direct product of sets:  $W(M)$  is not, in general, a group.)

*Proof.* We first prove that  $wMw^{-1}$  is again contained in the Siegel Levi.

The Levi  $M$  determines an equivalence relation  $\sim$  on the set of indices,  $\{1, \dots, n\}$  defined by the condition that  $i \sim i+1$  iff  $e_i - e_{i+1}$  is an root of  $M$ . View  $w$  of  $W(M)$  as a permutation of  $\{1, \dots, 2n\}$ . Because  $w$  is of minimal length,  $i \sim j$ ,  $i < j \Rightarrow w(i) < w(j)$ . Because  $wMw^{-1}$  is a standard Levi, we may deduce that if  $i \sim i+1$  then  $w(i+1) = w(i) + 1$ , except possibly when  $w(i) = n-1$ , in which case  $w(i+1)$  could, *a priori* be  $n+1$ . However, it is easy to check that in the special case when all  $m_i$  are even, the condition  $\det w = 1$  forces  $w(i+1) = w(i) + 1$  even if  $w(i) = n-1$ . It follows that  $wMw^{-1}$  is contained in the Siegel Levi.

When viewed as elements of  $\mathfrak{S}_n \times \{\pm 1\}^{n-1}$ , the elements of  $W(M)$  are those pairs  $(p, \underline{\epsilon})$  such that  $i \sim j \Rightarrow \epsilon_i = \epsilon_j$ , and  $i \sim i+1 \Rightarrow p(i+1) = p(i) + \epsilon_i$ . This gives the identification with  $\mathfrak{S}_r \times \{\pm 1\}^r$ .

It is clear that the precise value of  $g'_i$  is determined only up to conjugacy by an element of the torus (because we do not specify a particular representative for our Weyl group element). By Theorem 16.3.2 of [Spr], it may be discerned, to this level of precision, by looking at the effect of  $w$  on the based root datum of  $M$ . The result now follows from Lemma 6.2.3.  $\square$

**Corollary 6.2.6.** *Let  $w \in W(M)$  be associated to  $(p, \underline{\epsilon}) \in \mathfrak{S}_r \times \{\pm 1\}^r$  as above. Let  $\tau_1, \dots, \tau_r$  be irreducible cuspidal representations of  $GL_{m_1}(\mathbb{A}), \dots, GL_{m_r}(\mathbb{A})$ , respectively, and let  $\omega$  be a Hecke character. Then our identification of  $M$  with  $GL_{m_1} \times \dots \times GL_{m_r} \times GL_1$  determines an identification of  $\bigotimes_{i=1}^r \tau_i \boxtimes \omega$  with a representation of  $M(\mathbb{A})$ . Let  $M' = wMw^{-1}$ . Then  $M'$  is also identified, via 6.1.1 with  $GL_{m_{p^{-1}(1)}} \times \dots \times GL_{m_{p^{-1}(r)}} \times GL_1$ , and we have*

$$\bigotimes_{i=1}^r \tau_i \boxtimes \omega \circ Ad(w^{-1}) = \bigotimes_{i=1}^r \tau'_i \boxtimes \omega,$$

where

$$\tau'_i = \begin{cases} \tau_{p^{-1}(i)} & \text{if } \epsilon_{p^{-1}(i)} = 1, \\ \tilde{\tau}_{p^{-1}(i)} \otimes \omega & \text{if } \epsilon_{p^{-1}(i)} = -1. \end{cases}$$

*Proof.* The contragredient  $\tilde{\tau}_i$  of  $\tau_i$  may be realized as an action on the same space of functions as  $\tau_i$  via  $g \cdot \varphi(g_1) = \varphi(g_1 t g^{-1})$ . This follows from strong multiplicity one and the analogous statement for local representations, for which see [GK75] page 96, or [BZ1] page 57. Combining this fact with the Lemma, we obtain the Corollary.  $\square$

## 7. UNRAMIFIED CORRESPONDENCE

**Lemma 7.0.7.** *Suppose that  $\tau \cong \otimes'_v \tau_v$  is an  $\bar{\omega}$ -symplectic irreducible cuspidal automorphic representation of  $GL_{2n}(\mathbb{A})$ . Let  $v$  be a place such that  $\tau_v$  is unramified. Let  $t_{\tau,v}$  denote the semisimple conjugacy class in  $GL_{2n}(\mathbb{C})$  associated to  $\tau_v$ . Let  $r : GSp_{2n}(\mathbb{C}) \rightarrow GL_{2n}(\mathbb{C})$  be the natural inclusion. Then  $t_{\tau,v}$  contains elements of the image of  $r$ .*

*Proof.* For convenience in the application, we take  $GL_{2n}$  to be identified with a subgroup of the Levi of the Siegel parabolic as in section 6.1. Since  $\tau_v$  is both unramified and generic, it is isomorphic to  $\text{Ind}_{B(GL_{2n})(F_v)}^{GL_{2n}(F_v)} \mu$  for some unramified character  $\mu$  of the maximal torus  $T(GL_{2n})(F_v)$  such that this induced representation is irreducible. (See [Car], section 4, [Z] Theorem 8.1, p. 195.) Let  $\mu_i = \mu \circ e_i^*$ .

Since  $\tau \cong \tilde{\tau} \otimes \omega$ , it follows that  $\tau_v \cong \tilde{\tau}_v \otimes \omega_v$  and from this we deduce that  $\{\mu_i : 1 \leq i \leq 2n\}$  and  $\{\mu_i^{-1} \omega : 1 \leq i \leq 2n\}$  are the same set.

By Theorem 1, p. 213 of [Ja-Sh1], we have  $\prod_{i=1}^{2n} \mu_i = \omega^n$ .

Now, what we need to prove is the following: if  $S$  is a set of  $2n$  unramified characters of  $F_v$ , such that

- (1)  $\prod_{i=1}^{2n} \mu_i = \omega^n$
- (2) For each  $i$  there exists  $j$  such that  $\mu_i = \mu_j^{-1} \omega$

then there is a permutation  $\sigma : \{1, \dots, 2n\} \rightarrow \{1, \dots, 2n\}$  such that  $\mu_{\sigma(i)} = \omega \mu_{2n-\sigma(i)}^{-1}$  for  $i = 1$  to  $n$ . This we prove by induction on  $n$ . When  $n = 1$ , we know that  $\mu_1 = \mu_2^{-1} \omega$  for  $i = 1$  or  $2$ . If  $i = 2$  we are done, while if  $i = 1$  we use  $\omega = \mu_1 \mu_2$  to obtain  $\mu_1 = \mu_2$ , and the desired assertion. Now, if  $n > 1$  it is sufficient to show that there exist  $i \neq j$  such that  $\mu_i = \mu_j^{-1} \omega$ . If there exists  $i$  such that  $\mu_i \neq \mu_i^{-1} \omega$  then this is clear. On the other hand, there are exactly two unramified characters  $\mu$  such that  $\mu = \mu^{-1} \omega$ . The result follows  $\square$

The above argument easily yields:

**Corollary 7.0.8.** *Suppose  $\tau = \tau_1 \boxplus \dots \boxplus \tau_r$  with  $\tau_i$  an  $\bar{\omega}$ -symplectic irreducible cuspidal automorphic representation of  $GL_{2n_i}(\mathbb{A})$ , for each  $i$ . Then the same conclusion holds.*

## 8. EISENSTEIN SERIES I: CONSTRUCTION AND MAIN STATEMENTS

The main purpose of this section is to construct, for each integer  $n \geq 2$  and Hecke character  $\omega$ , a map from the set of all isobaric representations  $\tau$  satisfying the hypotheses of theorem 6 into the residual spectrum of  $G_{4n}$ . We use the same notation  $\mathcal{E}_{-1}(\tau, \omega)$  for all  $n$ . The construction is given by a multi-residue of an Eisenstein series in several complex variables, induced from the cuspidal representations  $\tau_1, \dots, \tau_r$  used to form  $\tau$ . (Note that by [Ja-Sh3], Theorem 4.4, p.809, this data is recoverable from  $\tau$ .)

Let  $\omega$  be a Hecke character. Let  $\tau_1, \dots, \tau_r$  be a irreducible cuspidal automorphic representations of  $GL_{2n_1}, \dots, GL_{2n_r}$ , respectively.

For each  $i$ , let  $V_{\tau_i}$  denote the space of cuspforms on which  $\tau_i$  acts. Then pointwise multiplication

$$\varphi_1 \otimes \dots \otimes \varphi_r \mapsto \prod_{i=1}^r \varphi_i$$

extends to an isomorphism between the abstract tensor product  $\bigotimes_{i=1}^r V_{\tau_i}$  and the space of all functions

$$\Phi(g_1, \dots, g_r) = \sum_{i=1}^N c_i \prod_{j=1}^r \varphi_{i,j}(g_j) \quad c_i \in \mathbb{C}, \varphi_{i,j} \in V_{\tau_j} \quad \forall i, j.$$

(This is an elementary exercise.) We consider the representation  $\tau_1 \otimes \dots \otimes \tau_r$  of  $GL_{2n_1} \times \dots \times GL_{2n_r}$ , realized on this latter space, which we denote  $V_{\otimes \tau_i}$ .

Let  $n = n_1 + \dots + n_r$ .

We will construct an Eisenstein series on  $G_{4n}$  induced from the subgroup  $P = MU$  of the Siegel parabolic such that  $M \cong GL_{2n_1} \times \dots \times GL_{2n_r} \times GL_1$ . Let  $s_1, \dots, s_r$  be a complex variables. Using the identification of  $M$  with  $GL_{2n_1} \times \dots \times GL_{2n_r} \times GL_1$  fixed in section 6.1 above, we define an action of  $M(\mathbb{A})$  on the space of  $\tau_1 \otimes \dots \otimes \tau_r$  by

$$(8.0.9) \quad (g_1, \dots, g_r, \alpha) \cdot \prod_{j=1}^r \varphi_j(h_j) = \left( \prod_{j=1}^r \varphi(h_j g_j) |\det g_j|^{s_j} \right) \omega(\alpha).$$

We denote this representation of  $M(\mathbb{A})$ , by  $(\bigotimes_{i=1}^r \tau_i \otimes |\det \cdot|^{s_i}) \boxtimes \omega$ . (We use  $\boxtimes$  to distinguish the “outer” tensor product with  $\omega$  from the “inner” tensor product with  $\det \cdot|^{s_i}$ . Recall that if  $V_1, V_2$  are two representations of the same group  $G$ , then the “outer” tensor product  $V_1 \boxtimes V_2$  is the representation of  $G \times G$  on the tensor product of the two spaces, while the “inner” tensor product  $V_1 \otimes V_2$  is the representation of  $G$  on the same space, acting diagonally.)

To shorten the notation, we write  $\underline{g} = (g_1, \dots, g_r)$ . Then (8.0.9) may be shortened to

$$(\underline{g}, \alpha) \cdot \Phi(\underline{h}) = \Phi(\underline{h} \cdot \underline{g}) \left( \prod_{j=1}^r |\det g_j|^{s_j} \right) \omega(\alpha).$$

We shall also employ the shorthand  $\underline{s} = (s_1, \dots, s_r)$ , and  $\underline{\tau} = (\tau_1, \dots, \tau_r)$ .

For each  $\underline{s}$  we have the induced representation  $\text{Ind}_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})} (\bigotimes_{i=1}^r \tau_i \otimes |\det_i|^{s_i}) \boxtimes \omega$ , (normalized induction) of  $G_{4n}(\mathbb{A})$ . The standard realization of this representation is action by right translation on the space  $V^{(1)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$  given by

$$\left\{ \tilde{F} : G_{4n}(\mathbb{A}) \rightarrow V_{\tau}, \text{ smooth } \left| \tilde{F}((\underline{g}, \alpha)h)(\underline{g}') = \tilde{F}(h)(\underline{g}'\underline{g})\omega(\alpha)|\delta_P|^{\frac{1}{2}} \prod_{i=1}^r |\det g_i|^{s_i} \right. \right\}.$$

Where

$$(8.0.10) \quad |\delta_P|^{\frac{1}{2}} = \prod_{i=1}^r |\det g_i|^{n - \frac{1}{2} + \sum_{j=i+1}^r n_j - \sum_{j=1}^{i-1} n_j}$$

makes the induction normalized.

A second useful realization is action by right translation on

$$V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega) = \left\{ f : G_{4n}(\mathbb{A}) \rightarrow \mathbb{C}, \left| f(h) = \tilde{F}(h)(e), \tilde{F} \in V^{(1)}(\underline{s}, \underline{\tau}, \omega) \right. \right\}.$$

Where  $e \in GL_{2n}(\mathbb{A})$  is the identity.

These vector spaces fit together into a vector bundle over  $\mathbb{C}^r$ . So a section of this bundle is a function  $f$  defined on  $\mathbb{C}^r$  such that  $f(\underline{s}) \in V^{(i)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$  ( $i = 1$  or  $2$ ) for each  $\underline{s}$ . Fix a maximal compact subgroup  $K$  of  $G_{4n}(\mathbb{A})$  satisfying the conditions required in [MW1] (see pages 1 and 4). Intersecting  $K$  with  $M(\mathbb{A})$  for a standard Levi  $M \subset G_{4n}$ , we fix maximal compact subgroups of these groups as well. We shall only require the use of flat,  $K$ -finite sections, which are defined as follows. Take  $f_0 \in V^{(i)}(\underline{0}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$   $K$ -finite, and define  $f(\underline{s})(h)$  by

$$f(\underline{s})(u(\underline{g}, \alpha)k) = f_0(u(\underline{g}, \alpha)k) \prod_{i=1}^r |\det g_i|^{s_i}$$

for  $u \in U(\mathbb{A})$ ,  $\underline{g} \in GL_{2n_1}(\mathbb{A}) \times \dots \times GL_{2n_r}(\mathbb{A})$ ,  $\alpha \in \mathbb{A}^\times$ ,  $k \in K$ . This is well defined. (I.e., although  $g_i$  is not uniquely determined in the decomposition,  $|\det g_i|$  is. Cf. the definition of  $m_P$  on p.7 of [MW1].)

We begin with a flat  $K$  finite section of the bundle of representations realized on the spaces  $V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ .

**Remark 8.0.11.** Clearly, the function  $f$  is determined by  $f(\underline{s}^*)$  for any choice of base point  $\underline{s}^*$ . In particular, any function of  $f$  may be regarded as a function of  $f_{\underline{s}^*} \in V^{(2)}(\underline{s}^*, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ , for any particular value of  $\underline{s}^*$ . We have exploited this fact with  $\underline{s}^* = 0$  to streamline the definitions. A posteriori it will become clear that the point  $\underline{s}^* = \frac{1}{2} := (\frac{1}{2}, \dots, \frac{1}{2})$  is of particular importance, and we shall then switch to  $\underline{s}^* = \frac{1}{2}$ .

For such  $f$  the sum

$$E(f)(g)(\underline{s}) := \sum_{\gamma \in P(F) \backslash G(F)} f(\underline{s})(\gamma g)$$

converges for all  $\underline{s}$  such that  $\text{Re}(s_r), \text{Re}(s_i - s_{i+1}), i = 1$  to  $r - 1$  are all sufficiently large. ([MW1], §II.1.5, pp.85-86). It has meromorphic continuation to  $\mathbb{C}^r$  ([MW1] §IV.1.8(a), IV.1.9(c), p.140).

These are our Eisenstein series. We collect some of their well-known properties in the following theorem.

**Theorem 8.0.12.** *We have the following:*

(1) *The function*

$$(8.0.13) \quad \prod_{i \neq j} (s_i + s_j - 1) \prod_{i=1}^r (s_i - \frac{1}{2}) E(f)(g)(\underline{s})$$

*is holomorphic at  $s = \frac{1}{2}$ . (More precisely, while  $E(f)(g)$  may have singularities, there is a holomorphic function defined on an open neighborhood of  $\underline{s} = \frac{1}{2}$  which agrees with (8.0.15) on the complement of the hyperplanes  $s_i = \frac{1}{2}$ , and  $s_i + s_j = 1$ .)*

(2) *The function (8.0.13) remains holomorphic (in the same sense) when  $s_i + s_j - 1$  is omitted, provided  $\tau_i \not\cong \omega \otimes \tilde{\tau}_j$ . It remains holomorphic when  $s_i - \frac{1}{2}$  is omitted, provided  $\tau_i$  is not  $\bar{\omega}$ -symplectic. Furthermore, each of these sufficient conditions is also necessary, in that the holomorphicity conclusion will fail, for some  $f$  and  $g$ , if any of the factors is omitted without the corresponding condition on  $\underline{s}$  being satisfied. From this we deduce that if*

(8.0.14) *the representations  $\tau_1, \dots, \tau_r$  are all distinct and  $\bar{\omega}$ -symplectic, then the function*

$$(8.0.15) \quad \prod_{i=1}^r (s_i - \frac{1}{2}) E(f)(g)(\underline{s})$$

*is holomorphic at  $s = \frac{1}{2}$  for all  $f, g$  and nonvanishing at  $s = \frac{1}{2}$  for some  $f, g$ .*

(3) *Let us now assume condition (8.0.14) holds, and regard  $f$  as a function of  $f_{\frac{1}{2}} \in V^{(2)}(\frac{1}{2}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ . Let  $E_{-1}(f_{\frac{1}{2}})(g)$  denote the value of the function (8.0.15) at  $\underline{s} = \frac{1}{2}$  (defined by analytic continuation). Then  $E_{-1}(f)$  is an  $L^2$  function for all  $f_{\frac{1}{2}} \in V^{(2)}(\frac{1}{2}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ .*

(4) *The function  $E_{-1}$  is an intertwining operator from  $\text{Ind}_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i \otimes |\det_i|^{\frac{1}{2}}) \boxtimes \omega$  into the space of  $L^2$  automorphic forms.*

(5) *If  $\mathcal{E}_{-1}(\tau, \omega)$  is the image of  $E_{-1}$ , and  $\psi_{LW}$  is the character of  $U_{\max}$  given by  $\psi_{LW}(u) = \psi_0(\sum_{i=1}^{2n-1} u_{i,i+1})$ , then  $(U_{\max}, \psi_{LW}) \notin \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$ .*

(6) *The space of functions  $\mathcal{E}_{-1}(\tau, \omega)$  does not depend on the order chosen on the cuspidal representations  $\tau_1, \dots, \tau_r$ . Thus it is well-defined as a function of the isobaric representation  $\tau$ .*

**Remark 8.0.16.** *By induction in stages, the induced representation  $\text{Ind}_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i \otimes |\det_i|^{\frac{1}{2}}) \boxtimes \omega$ , which comes up in part (4) of the theorem can also be written as  $\text{Ind}_{P_{\text{Siegel}}(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$ , where  $\tau = \tau_1 \boxplus \dots \boxplus \tau_r$  as before, and  $P_{\text{Siegel}}$  is the Siegel parabolic. (Cf. section 2.4.) Here, we also exploit the identification of the Levi  $M_{\text{Siegel}}$  of  $P_{\text{Siegel}}$  with  $GL_{2n} \times GL_1$  fixed in 6.1.1.*

*Proof.* The proof is virtually identical to the proof of theorem 15.0.12. In two places the proof of theorem 15.0.12 is slightly more complicated, and therefore we include complete details for that case, and for this case only describe the differences.

One must change “ $4n + 1$ ” to “ $4n$ ,” obviously, and one must change “ $\omega^{-1}$ -orthogonal” to “ $\omega^{-1}$ -symplectic.” The twisted exterior square  $L$  function plays the role of the twisted symmetric square  $L$  function. The expression for  $|\delta_P|^{\frac{1}{2}}$  is (8.0.10) instead of (15.0.10). The rational character  $\varepsilon_i$ , ( $1 \leq i \leq r$ ) as in (17.2.1) is no longer the restriction of a positive root, and therefore every restricted

root is indivisible. This simplifies various statments. Finally, the analogue of remark 18.0.5 is simpler, since a representation of  $GL_m$  can be  $\omega^{-1}$ -symplectic only if  $m$  is even (with no condition on  $\omega$ ).  $\square$

## 9. DESCENT CONSTRUCTION

**9.1. Vanishing of deeper descents and the descent representation.** In this section, we shall make use of remark 8.0.16, and regard  $\mathcal{E}_{-1}(\tau, \omega)$  as affording an automorphic realization of the representation induced from the representation  $\tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$  of the Siegel Levi. Thus we may dispense with the smaller Levi denoted by  $P$  in the previous section, and in this section we denote the Siegel parabolic more briefly by  $P = MU$ .

Next we describe certain unipotent periods of  $G_{2m}$  which play a key role in the argument. For  $1 \leq \ell < m$ , let  $N_\ell$  be the subgroup of  $U_{\max}$  defined by  $u_{ij} = 0$  for  $i > \ell$ . (Recall that according to the convention above, this refers only to those  $i, j$  with  $i < j \leq m - i$ .) This is the unipotent radical of a standard parabolic  $Q_\ell$  having Levi  $L_\ell$  isomorphic to  $GL_1^\ell \times G_{2m-2\ell}$ .

Let  $\vartheta$  be a character of  $N_\ell$  then we may define

$$DC^\ell(\tau, \omega, \vartheta) = FC^\vartheta \mathcal{E}_{-1}(\tau, \omega).$$

**Theorem 9.1.1.** *Let  $\omega$  be a Hecke character. Let  $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$  be an isobaric sum of  $\bar{\omega}$ -symplectic irreducible cuspidal automorphic representations  $\tau_1, \dots, \tau_r$ , of  $GL_{2n_1}(\mathbb{A}), \dots, GL_{2n_r}(\mathbb{A})$ , respectively. If  $\ell \geq n$ , and  $\vartheta$  is in general position, then*

$$DC^\ell(\tau, \omega, \vartheta) = \{0\}.$$

*Proof.* By Theorem 8.0.12, (3) the representation  $\mathcal{E}_{-1}(\tau, \omega)$  decomposes discretely. Let  $\pi \cong \otimes'_v \pi_v$  be one of the irreducible components, and  $p_\pi : \mathcal{E}_{-1}(\tau, \omega) \rightarrow \pi$  the natural projection.

Fix a place  $v_0$  such which  $\tau_{v_0}$  and  $\pi_{v_0}$  are unramified. For any  $\xi^{v_0} \in \otimes'_{v \neq v_0} Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|^{\frac{1}{2}} \boxtimes \omega_v$  we define a map

$$i_{\xi^{v_0}} : Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|^{\frac{1}{2}} \boxtimes \omega_{v_0} \rightarrow Ind_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$$

by  $i_{\xi^{v_0}}(\xi_v) = \iota(\xi_{v_0} \otimes \xi^{v_0})$ , where  $\iota$  is an isomorphism of the restricted product  $\otimes'_v Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|^{\frac{1}{2}} \boxtimes \omega_v$  with the global induced representation  $Ind_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$ . Clearly

$$\mathcal{E}_{-1}(\tau, \omega) = E_{-1} \circ \iota(\otimes'_v Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|^{\frac{1}{2}} \boxtimes \omega_v).$$

For any decomposable vector  $\xi = \xi_{v_0} \otimes \xi^{v_0}$ ,

$$p_\pi \circ E_{-1} \circ \iota(\xi) = p_\pi \circ E_{-1} \circ i_{\xi^{v_0}}(\xi_{v_0}).$$

Thus,  $\pi_{v_0}$  is a quotient of  $Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|^{\frac{1}{2}} \boxtimes \omega_{v_0}$ , and hence (since we took  $v_0$  such that  $\pi_{v_0}$  is unramified) it is isomorphic to the unramified constituent  ${}^{un}Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|^{\frac{1}{2}} \boxtimes \omega_{v_0}$ .

Denote the isomorphism of  $\pi$  with  $\otimes'_v \pi_v$  by the same symbol  $\iota$ . This time, fix  $\zeta^{v_0} \in \otimes'_{v \neq v_0} \pi_v$ , and define  $i_{\zeta^{v_0}} : {}^{un}Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|^{\frac{1}{2}} \boxtimes \omega_{v_0} \rightarrow \pi$ . It follows easily from the definitions that

$$FC^\vartheta \circ i_{\zeta^{v_0}}$$

factors through the Jacquet module  $\mathcal{J}_{N_\ell, \vartheta}({}^{un}Ind_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|^{\frac{1}{2}} \boxtimes \omega_{v_0})$ . In appendix 10 we show that this Jacquet module is zero. The result follows.  $\square$



**Remark 9.1.2.** A general character of  $N_\ell$  is of the form

$$\psi_0(c_1 u_{1,2} + \cdots + c_{\ell-1} u_{\ell-1,\ell} + d_1 u_{\ell,\ell+1} + \cdots + d_{4n-2\ell} u_{\ell,4n-\ell}).$$

The Levi  $L_\ell$  acts on the space of characters (cf. section 5). Over an algebraically closed field there is an open orbit, which consists of all those elements such that  $c_i \neq 0$  for all  $i$  and  ${}^t \underline{d} J \underline{d} \neq 0$ . Here,  $\underline{d}$  is the column vector  ${}^t(d_1, \dots, d_{4n-2\ell})$ , and  $J$  is defined as in 3.1. Over a general field two such elements are in the same  $F$ -orbit iff the two values of  ${}^t \underline{d} J \underline{d}$  are in the same square class.

Let  $\Psi_\ell$  be the character of  $N_\ell$  defined by

$$\Psi_\ell(u) = \psi_0(u_{12} + \cdots + u_{\ell-1,\ell} + u_{\ell,2n} - u_{\ell,2n+1}).$$

It is not hard to see that

- the stabilizer  $L_\ell^{\Psi_\ell}$  (cf.  $M^\vartheta$  in definition 5.0.2) has two connected components,
- the one containing the identity is isomorphic to  $G_{4n-2\ell-1}$ ,
- there is an “obvious” choice of isomorphism  $inc : G_{4n-2\ell-1} \rightarrow (L_\ell^{\Psi_\ell})^0$  having the following property: if  $\{e_i^* : i = 0 \text{ to } 2n\}$  is the basis for the cocharacter lattice of  $G_{4n}$  as in section 4.1, and  $\{\bar{e}_i^*, i = 0 \text{ to } 2n - \ell - 1\}$  is the basis for that of  $G_{4n-2\ell-1}$ , then

$$(9.1.3) \quad inc \circ \bar{e}_i^* = \begin{cases} e_0^*, & i = 0 \\ e_{\ell+i}^*, & i = 1 \text{ to } 2n - \ell - 1. \end{cases}$$

In the case when  $\ell = 2n - 1$ ,  $N_\ell = U_{\max}$ , and  $\Psi_\ell$  is a generic character. The above remarks remain valid with the convention that  $G_1 = GL_1$ .

Let

$$DC_\omega(\tau) = FC^{\Psi_{n-1}} \mathcal{E}_{-1}(\tau, \omega).$$

It is a space of smooth functions  $G_{2n+1}(F \backslash \mathbb{A}) \rightarrow \mathbb{C}$ , and affords a representation of the group  $G_{2n+1}(\mathbb{A})$  acting by right translation, where we have identified  $G_{2n+1}$  with the identity component of  $L_{n-1}^{\Psi_{n-1}}$ .

## 9.2. Main Result.

**Theorem 9.2.1.** Let  $\omega$  be a Hecke character. Let  $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$  be an isobaric sum of distinct  $\bar{\omega}$ -symplectic irreducible cuspidal automorphic representations  $\tau_1, \dots, \tau_r$ , of  $GL_{2n_1}(\mathbb{A}), \dots, GL_{2n_r}(\mathbb{A})$ , respectively.

- (1) The space  $DC_\omega(\tau)$  is a nonzero cuspidal representation of  $G_{2n+1}(\mathbb{A})$ . Furthermore, the representation  $DC_\omega(\tau)$  supports a nonzero Whittaker integral.
- (2) If  $\sigma$  is any irreducible automorphic representation contained in  $DC_\omega(\tau)$ , then  $\sigma$  lifts weakly to  $\tau$  under the map  $r$ . Also, the central character of  $\sigma$  is  $\omega$ .

**Remark 9.2.2.** Since  $DC_\omega(\tau)$  is nonzero and cuspidal, there exists at least one irreducible component  $\sigma$ . In the case of orthogonal groups, one may show ([So1], pp. 342, item 4) that all of the components are generic using the Rankin-Selberg integrals of [Gi-PS-R], [So2]. On the other hand, in the odd case, one may also show ([GRS4], Theorem 8, p. 757, or [So1] page 342, item 6) using the results of [Ji-So] that  $DC_\omega(\tau)$  is irreducible. The extension of [Ji-So] to  $GSpin$  groups is a work in progress of Takeda and Lau.

## 9.3. Proof of main theorem.

*Proof.* The statements are proved by combining relationships between unipotent periods and knowledge about  $\mathcal{E}_{-1}(\tau, \omega)$ .

- (1) **Nonvanishing and genericity** For genericity, let  $(U_1, \psi_1)$  denote the unipotent period obtained by composing the one which defines the descent with the one which defines the Whittaker function on  $G_{2n+1}$  embedded into  $G_{4n}$  as the stabilizer of the descent character. Thus  $U_1$  is the subgroup of the standard maximal unipotent defined by the relations  $u_{i,2n} = u_{i,2n+1}$  for  $i = n$  to  $2n-1$ , and

$$\psi_1(u) = \psi_0(u_{1,2} + \cdots + u_{n-2,n-1} + u_{n-1,2n} - u_{n-1,2n+1} + u_{n,n+1} + \cdots + u_{2n-1,2n}).$$

Next, let  $U_2$  denote the subgroup of the standard maximal unipotent defined by  $u_{i,i+1} = 0$  for  $i$  even and less than  $2n$ . (One may also put  $\leq 2n$ : the equation  $u_{2n,2n+1} = 0$  is automatic for any element of  $U_{\max}$ .) The character  $\psi_2$  depends on whether  $n$  is odd or even. If  $n$  is even, it is

$$\psi_0(u_{1,3} + u_{2,4} + \cdots + u_{2n-1,2n+1}),$$

while, if  $n$  is odd, it is

$$\psi_0(u_{1,3} + u_{2,4} + \cdots + u_{2n-3,2n-1} + u_{2n-2,2n+1} + u_{2n-1,2n}),$$

Finally, let  $U_3$  denote the maximal unipotent, and  $\psi_3$  denote

$$\psi_3(u) = \psi_0(u_{1,2} + \cdots + u_{2n-1,2n}).$$

Thus  $(U_3, \psi_3)$  is the composite of the unipotent period defining the constant term along the Siegel parabolic, and the one which defines the Whittaker functional on the Levi of this parabolic. By Theorem 8.0.12 (5) this period is *not* in  $\mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$ .

In the appendices, we show

- (a)  $(U_1, \psi_1)|(U_2, \psi_2)$ , in Lemma 11.2.1, and
- (b)  $(U_3, \psi_3) \in \langle (U_2, \psi_2), \{(N_\ell, \vartheta) : n \leq \ell < 2n \text{ and } \vartheta \text{ in general position.}\} \rangle$  in Lemma 11.2.2.

By Theorem 9.1.1  $(N_\ell, \vartheta) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$  for all  $n \leq \ell < 2n$  and  $\vartheta$  in general position. It follows that  $(U_1, \psi_1) \notin \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$ . This establishes genericity (and hence nontriviality) of the descent.

- (2) **Cuspidality** Turning to cuspidality, we prove in the appendices an identity relating:

- Constant terms on  $G_{2n+1}$  embedded as  $(L_{n-1}^{\Psi_{n-1}})^0$ ,
- Descent periods in  $G_{4n}$ ,
- Constant terms on  $G_{4n}$ ,
- Descent periods on  $G_{4n-2k}$ , embedded in  $G_{4n}$  as a subgroup of a Levi.

To formulate the exact relationship we introduce some notation for the maximal parabolics of GSpin groups.

The group  $G_{2n+1}$  has one standard maximal parabolic having Levi  $GL_i \times G_{2n-2i+1}$  for each value of  $i$  from 1 to  $n$ . Let us denote the unipotent radical of this parabolic by  $V_i^{2n+1}$ . We denote the trivial character of any unipotent group by  $\mathbf{1}$ .

The group  $G_{4n}$  has one standard maximal parabolic having Levi  $GL_k \times G_{4n-2k}$  for each value of  $k$  from 1 to  $2n-2$ . We denote the unipotent radical of this parabolic by  $V_k$ .

(The group  $G_{4n}$  also has two parabolics with Levi isomorphic to  $GL_{2n} \times GL_1$ , but since they will not come up in this discussion, we do not need to bother over a notation to distinguish them.)

We prove in Lemma 11.2.4 that  $(V_k^{2n+1}, \mathbf{1}) \circ (N_{n-1}, \Psi_{n-1})$  is contained in

$$\langle (N_{n+k-1}, \Psi_{n+k-1}), \{(N_{n+j-1}, \Psi_{n+j-1})^{(4n-2k+2j)} \circ (V_{k-j}, \mathbf{1}) : 1 \leq j < k\} \rangle,$$

where  $(N_{n+j-1}, \Psi_{n+j-1})^{(4n-2k+2j)}$  denotes the descent period, defined as above, but on the group  $G_{4n-2k+2j}$ , embedded into  $G_{4n}$  as a component of the Levi with unipotent radical  $V_{k-j}$ .

By Theorem 9.1.1  $(N_{n+k-1}, \Psi_{n+k-1}) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$  for  $k = 1$  to  $n$ . Furthermore, for  $k, j$  such that  $1 \leq j < k \leq n$ , the function  $E(f)(s)^{(V_{k-j}, \mathbf{1})}$  may be expressed in terms of Eisenstein series on  $GL_{k-j}$  and  $G_{4n-2k+2j}$  using Proposition II.1.7 (ii) of [MW1]. What we require is the following:

**Lemma 9.3.1.** *For all  $f \in V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau \boxtimes \omega)$*

$$E_{-1}(f)^{(V_{k-j}, \mathbf{1})} \Big|_{G_{4n-2k+2j}(\mathbb{A})} \in \bigoplus_S \mathcal{E}_{-1}(\tau_S, \omega),$$

where the sum is over subsets  $S$  of  $\{1, \dots, r\}$  such that  $\sum_{i \in S} 2n_i = 2n - k + j$ , and, for each such  $S$ ,  $\mathcal{E}_{-1}(\tau_S, \omega)$  is the space of functions on  $G_{4n-2k+2j}(\mathbb{A})$  obtained by applying the construction of  $\mathcal{E}_{-1}(\tau, \omega)$  to  $\{\tau_i : i \in S\}$ , instead of  $\{\tau_i : 1 \leq i \leq r\}$ .

Once again, this is immediate from [MW1] Proposition II.1.7 (ii).

Applying Theorem 9.1.1, with  $\tau$  replaced by  $\tau_S$  and  $2n$  by  $2n - k + j$ , we deduce

$$(N_{n+j-1}, \Psi_{n+j-1})^{(4n-2k+2j)} \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau_S, \omega)) \quad \forall S,$$

and hence  $(N_{n+j-1}, \Psi_{n+j-1})^{(4n-2k+2j)} \circ (V_{k-j}, \mathbf{1}) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$ . This shows that any nonzero function appearing in the space  $DC_\omega(\tau)$  must be cuspidal. Such a function is also easily seen to be of uniformly moderate growth, being the integral of an automorphic form over a compact domain. In addition, such a function is easily seen to have central character  $\omega$ , and any function with these properties is necessarily square integrable modulo the center ([MW1] I.2.12). It follows that the space  $DC_\omega(\tau)$  decomposes discretely.

(3) **The unramified parameters of descent:**

Now, suppose  $\sigma \cong \otimes'_v \sigma_v$  is an irreducible representation which is a constituent of  $DC_\omega(\tau)$ . Let  $p_\sigma : DC_\omega(\tau) \rightarrow \sigma$  be the natural projection.

Once again, by Theorem 8.0.12, (3) the representation  $\mathcal{E}_{-1}(\tau, \omega)$  decomposes discretely. Let  $\pi$  be an irreducible component of  $\mathcal{E}_{-1}(\tau, \omega)$  such that the restriction of  $p_\sigma \circ FC^{\Psi_{n-1}}$  to  $\pi$  is nontrivial. As discussed previously in the proof of Theorem 9.1.1, at all but finitely many  $v$ ,  $\tau$  is unramified at  $v$  and furthermore,  $\pi_v$  is the unramified constituent  ${}^{un}Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \boxtimes \omega_v \otimes$

$|\det|_v^{\frac{1}{2}}$  of  $Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \boxtimes \omega_v \otimes |\det|_v^{\frac{1}{2}}$ . If  $v_0$  is such a place, the map  $p_\sigma \circ FC^{\Psi_{n-1}} \circ i_{\zeta^{v_0}}$ , with  $i_{\zeta^{v_0}}$  defined as in Theorem 9.1.1, factors through  $\mathcal{J}_{N_{n-1}, \Psi_{n-1}} \left( {}^{un}Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v \right)$ , and gives rise to a  $G_{2n+1}(F_{v_0})$ -equivariant map from this Jacquet-module onto  $\sigma_{v_0}$ .

To pin things down precisely, assume that  $\tau_v$  is the unramified component of  $Ind_{B(GL_{2n})(F_v)}^{GL_{2n}(F_v)} \mu$ , and let  $\mu_1, \dots, \mu_{2n}$  be defined as in the proof of Lemma 7.0.7. By Lemma 7.0.7, we may assume without loss of generality that  $\mu_{2n+1-i} = \omega \mu_i^{-1}$  for  $i = 1$  to  $n$ .

We also need to refer to the elements of the basis of the cocharacter lattice of  $G_{2n+1}$  fixed in section 4.1. As in the remarks preceding the definition of  $DC_\omega(\tau)$ , we denote these  $\bar{e}_0^*, \dots, \bar{e}_n^*$ .

In the appendices, we show that

$$\mathcal{J}_{N_{n-1}, \Psi_{n-1}} \left( {}^{un}Ind_{P(F_v)}^{G_{4n}(F_v)} \tau_v \boxtimes \omega_v \otimes |\det|_v^{\frac{1}{2}} \right)$$

is isomorphic as a  $G_{2n+1}(F_v)$ -module to  $Ind_{B(G_{2n+1})(F_v)}^{G_{2n+1}(F_v)} \chi$  for  $\chi$  the unramified character of  $B(G_{2n+1})(F_v)$  such that

$$\chi \circ \bar{e}_i^* = \mu_i, i = 1 \text{ to } n, \chi \circ \bar{e}_0^* = \omega_v.$$

It follows that  $\tau$  is a weak lift of  $\sigma$  associated to the map  $r$ .

□

## 10. APPENDIX I: LOCAL RESULTS ON JACQUET FUNCTORS

In this appendix,  $F$  is a non-archimedean local field, on which we place the additional technical hypothesis

$$(10.0.2) \quad B(G_{2n-1})(F)G_{2n-1}(\mathfrak{o}) = G_{2n-1}(F),$$

which is known (see [Tits], 3.9, and 3.3.2) to hold at all but finitely many non-Archimedean completions of a number field. Here,  $G_{2n-1}$  is identified with  $(L_{n-1}^\psi)^0$  is defined as in (9.1.3), and  $\mathfrak{o}$  denotes the ring of integers of  $F$ .

**Proposition 10.0.3.** *Let  $\tau = \text{Ind}_{B(GL_{2n})(F)}^{GL_{2n}(F)} \mu$ , where  $\mu$  satisfies  $\mu \circ e_i^* = \omega \mu \circ e_{2n+1-i}^*$ , and let  $P$  denote the Siegel parabolic subgroup. Then for  $\ell \geq n$  and  $\vartheta$  in general position, the Jacquet module  $\mathcal{J}_{N_\ell, \vartheta}(\text{un Ind}_{P(F)}^{G_{4n}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega)$  is trivial.*

*Proof.* First, let  $\mu_i : F \rightarrow \mathbb{C}$  be the unramified character given by  $\mu_i = \mu \circ e_i^*$ . By induction in stages,

$$\text{un Ind}_{P(F)}^{G_{4n}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega = \text{un Ind}_{B(G_{4n})(F)}^{G_{4n}(F)} \tilde{\mu},$$

where  $\tilde{\mu} \circ e_i^*(x) = |x|^{\frac{1}{2}} \mu_i(x)$ , for  $i = 1$  to  $2n$  and  $\tilde{\mu} \circ e_0^* = \omega$ . If  $\tilde{\mu}'$  is the character such that  $\tilde{\mu}' \circ e_{2i-1}^*(x) = \mu_i(x)|x|^{\frac{1}{2}}$ , and  $\tilde{\mu}' \circ e_{2i}^*(x) = \mu_i(x)|x|^{-\frac{1}{2}}$ , for  $i = 1$  to  $n$ , and  $\tilde{\mu}' \circ e_0^* = \omega$ , then it follows from lemma 6.2.3 that  $\tilde{\mu}'$  is in the Weyl orbit of  $\tilde{\mu}$ . Hence, by the definition of the unramified constituent

$$\text{un Ind}_{B(G_{4n})(F)}^{G_{4n}(F)} \tilde{\mu} = \text{un Ind}_{B(G_{4n})(F)}^{G_{4n}(F)} \tilde{\mu}'.$$

Now, it is well known that

$$\text{un Ind}_{B(GL_2)(F)}^{GL_2(F)} \mu | \cdot |^{\frac{1}{2}} \otimes \mu | \cdot |^{-\frac{1}{2}} = \mu \circ \det.$$

It follows that

$$\text{un Ind}_{B(G_{4n})(F)}^{G_{4n}(F)} \tilde{\mu}' = \text{un Ind}_{P_{2n}(F)}^{G_{4n}(F)} \hat{\mu},$$

where  $P_{2n}$  is the parabolic of  $G_{4n}$  having Levi isomorphic to  $GL_2^n \times GL_1$ , such that the roots of this Levi are  $e_1 - e_2, e_3 - e_4, \dots, e_{2n-1} - e_{2n}$ , and  $\hat{\mu}$  is the character given by  $\hat{\mu} \circ e_{2i-1}^* = \hat{\mu} \circ e_{2i}^* = \mu_i, \hat{\mu} \circ e_0^* = \omega$ .

The remainder of the proof of this lemma as well as the next proposition may be viewed as a detailed worked example of theorem 5.2 of [BZ2].

The space  $\text{Ind}_{P_{2n}(F)}^{G_{4n}(F)} \hat{\mu}$  has a filtration as a  $Q_\ell(F)$ -module, in terms of  $Q_\ell(F)$ -modules indexed by the elements of  $(W \cap P_1) \backslash W / (W \cap Q_\ell)$ . For any element  $x$  of  $P_1(F)wQ_\ell(F)$  the module corresponding to  $w$  is isomorphic to  $c - \text{ind}_{x^{-1}P_1(F)x \cap Q_\ell(F)}^{Q_\ell(F)} \hat{\mu} \delta_{P_1}^{\frac{1}{2}} \circ \text{Ad}(x)$ . Here  $\text{Ad}(x)$  denotes the map given by conjugation by  $x$ . It sends  $x^{-1}P_1(F)x \cap Q_\ell(F)$  into  $P_1(F)$ . Also, here and throughout  $c - \text{ind}$  denotes non-normalized compact induction. (See [Cass], section 6.3.)

Recall from 6.2 that the Weyl group of  $G_{4n}$  is identified (canonically after the choice of pr) with the set of permutations  $w \in \mathfrak{S}_{4n}$  satisfying,

- (1)  $w(4n+1-i) = 4n+1-w(i)$  and
- (2)  $\det w = 1$  when  $w$  is written as a  $4n \times 4n$  permutation matrix.

As representatives for the double cosets  $(W \cap P_1) \backslash W / (W \cap Q_\ell)$  we choose the element of minimal length in each. As permutations, these elements have the properties

- (3)  $w^{-1}(2i) > w^{-1}(2i-1)$  for  $i = 1$  to  $2n$ , and
- (4) If  $\ell \leq i < j \leq 4n+1-\ell$  and  $w(i) > w(j)$ , then  $i = 2n$  and  $j = 2n+1$ .

Let  $I_w$  be the  $Q_\ell(F)$ -module obtained as

$$c - \text{ind}_{\dot{w}^{-1}P_1(F)\dot{w}\cap Q_\ell(F)}^{Q_\ell(F)} \hat{\mu} \delta_{P_1}^{\frac{1}{2}} \circ \text{Ad}(\dot{w})$$

using any element  $\dot{w}$  of  $\text{pr}^{-1}(w)$ .

A function  $f$  in  $I_w$  will map to zero under the natural projection to  $\mathcal{J}_{N_\ell, \vartheta}(I_w)$  iff there exists a compact subgroup  $N_\ell^0$  of  $N_\ell(F)$  such that

$$\int_{N_\ell^0} f(hn) \overline{\vartheta(n)} dn = 0 \quad \forall h \in Q_\ell(F).$$

(See [Cass], section 3.2.) Let  $\vartheta^h(n) = \vartheta(hnh^{-1})$ . It is easy to see that the integral above vanishes for suitable  $N_\ell^0$  whenever

$$(10.0.4) \quad \vartheta^h|_{N_\ell(F) \cap w^{-1}P_{2n}(F)w} \text{ is nontrivial.}$$

Furthermore, the function  $h \mapsto \vartheta^h$  is continuous in  $h$ , (the topology on the space of characters of  $N_\ell(F)$  being defined by identifying it with a finite dimensional  $F$ -vector space, cf. section 5) so if this condition holds for all  $h$  in a compact set, then  $N_\ell^0$  can be made uniform in  $h$ .

Now,  $\vartheta$  is in general position. Hence, so is  $\vartheta^h$  for every  $h$ . So, if we write

$$\vartheta^h(u) = \psi_0(c_1 u_{1,2} + \cdots + c_{\ell-1} u_{\ell-1,\ell} + d_1 u_{\ell,\ell+1} + \cdots + d_{2m-2\ell} u_{\ell,2m-\ell}),$$

we have that  $c_i \neq 0$  for all  $i$  and  ${}^t \underline{d} J \underline{d} \neq 0$ .

Clearly, the condition (10.0.4) holds for all  $h$  unless

$$(5) \quad w(1) > w(2) > \cdots > w(\ell).$$

Furthermore, because  ${}^t \underline{d} J \underline{d} \neq 0$ , there exists some  $i_0$  with  $\ell + 1 \leq i_0 \leq 2n$  such that  $d_{i_0-\ell} \neq 0$  and  $d_{4n+1+\ell-i_0} \neq 0$ . From this we deduce that the condition (10.0.4) holds for all  $h$  unless  $w$  has the additional property

$$(6) \quad \text{There exists } i_0 \text{ such that } w(\ell) > w(i_0) \text{ and } w(\ell) > w(4n+1-i_0).$$

However, if  $\ell \geq n$  it is easy to check that no permutations with properties (1),(3) (5) and (6) exist.

Thus  $\mathcal{J}_{N_\ell, \vartheta}(I_w) = \{0\}$  for all  $w$  and hence  $\mathcal{J}_{N_\ell, \vartheta}({}^{un} \text{Ind}_{P(F)}^{G_{4n}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega) = \{0\}$  by exactness of the Jacquet functor.  $\square$

**Proposition 10.0.5.** *Let  $\tau = \text{Ind}_{B(GL_{2n})(F)}^{GL_{2n}(F)} \mu$ , where  $\mu$  satisfies  $\mu \circ e_i^* = \omega \mu \circ e_{2n+1-i}$ . Then the Jacquet module*

$$\mathcal{J}_{N_{n-1}, \Psi_{n-1}} \left( {}^{un} \text{Ind}_{P(F)}^{G_{4n}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega \right)$$

*is isomorphic as a  $G_{2n+1}(F)$ -module to a subquotient of  $\text{Ind}_{B(G_{2n+1})(F)}^{G_{2n+1}(F)} \chi$  for  $\chi$  the unramified character of  $B(G_{2n+1})(F)$  such that*

$$\chi \circ \bar{e}_i^* = \mu_i, i = 1 \text{ to } n, \chi \circ \bar{e}_0^* = \omega.$$

*Proof.* As before, we have

$${}^{un} \text{Ind}_{P(F)}^{G_{4n}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega = {}^{un} \text{Ind}_{P_{2n}}^{G_{4n}(F)} \hat{\mu},$$

and we filter  $\text{Ind}_{P_{2n}(F)}^{G_{4n}(F)} \hat{\mu}$  in terms of  $Q_{n-1}(F)$ -modules  $I_w$ . This time,  $\mathcal{J}_{N_{n-1}, \Psi_{n-1}}(I_w) = \{0\}$  for all  $w$  except one. This one Weyl element, which we denote  $w_0$ , corresponds to the unique permutation satisfying (1),(2),(3),(4) of the previous result, together with  $w(i) = 4n - 2i + 1$  for  $i = 1$  to  $n - 1$ . Exactness yields

$$\mathcal{J}_{N_{n-1}, \Psi_{n-1}} \left( {}^{un} \text{Ind}_{P(F)}^{G_{4n}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega \right) \cong \mathcal{J}_{N_{n-1}, \Psi_{n-1}}(I_{w_0}).$$

(This is an isomorphism of  $Q_{n-1}^{\Psi_{n-1}}(F)$ -modules, where  $Q_{n-1}^{\Psi_{n-1}} = N_{n-1} \cdot L_{n-1}^{\Psi_{n-1}} \subset Q_{n-1}$ , is the stabilizer of  $\Psi_{n-1}$  in  $Q_{n-1}$  (cf.  $L^\vartheta$  above).)

Now, recall that for each  $h \in Q_{n-1}(F)$  the character  $\Psi_{n-1}^h(u) = \Psi_{n-1}(huh^{-1})$  is a character of  $N_{n-1}$  in general position, and as such determines coefficients  $c_1, \dots, c_{n-2}$  and  $d_1, \dots, d_{2n+2}$  as in remark 9.1.2. Clearly,

$$Q_{n-1}^o := \{h \in Q_{n-1}(F) \mid d_i \neq 0 \text{ for some } i \neq n+1, n+2\}$$

is open. Moreover, one may see from the description of  $w_0$  that for  $h$  in this set 10.0.4 is satisfied. We have an exact sequence of  $Q_{n-1}^{\Psi_{n-1}}(F)$ -modules

$$0 \rightarrow I_{w_0}^* \rightarrow I_{w_0} \rightarrow \bar{I}_{w_0} \rightarrow 0,$$

where  $I_w^*$  consists of those functions in  $I_w$  whose compact support happens to be contained in  $Q_{n-1}^o$ , and the third arrow is restriction to the complement of  $Q_{n-1}^o$ . This complement is slightly larger than  $Q_{n-1}^{\Psi_{n-1}}(F)$  in that it contains the full torus of  $Q_{n-1}(F)$ , but restriction of functions is an isomorphism of  $Q_{n-1}^{\Psi_{n-1}}(F)$ -modules,

$$\bar{I}_{w_0} \rightarrow c - \text{ind}_{Q_{n-1}^{\Psi_{n-1}}(F) \cap w_0^{-1}P_{2n}(F)w_0}^{Q_{n-1}^{\Psi_{n-1}}(F)} \hat{\mu} \delta_{P_{2n}}^{\frac{1}{2}} \circ \text{Ad}(w_0).$$

Clearly  $\mathcal{J}_{N_{n-1}, \Psi_{n-1}}(I_{w_0}^*) = \{0\}$ , and hence

$$\mathcal{J}_{N_{n-1}, \Psi_{n-1}} \left( \text{Ind}_{P_{2n}(F)}^{G_{4n}(F)} \hat{\mu} \right) \cong \mathcal{J}_{N_{n-1}, \Psi_{n-1}} \left( c - \text{ind}_{Q_{n-1}^{\Psi_{n-1}}(F) \cap w_0^{-1}P_{2n}(F)w_0}^{Q_{n-1}^{\Psi_{n-1}}(F)} \hat{\mu} \delta_{P_{2n}}^{\frac{1}{2}} \circ \text{Ad}(w_0) \right).$$

Now let  $\mathcal{W}$  denote

$$\left\{ f : Q_{n-1}^{\Psi_{n-1}}(F) \rightarrow \mathbb{C} \mid \begin{array}{l} f(uq) = \Psi_{n-1}(u)f(q) \ \forall u \in N_{n-1}(F), \ q \in Q_{n-1}^{\Psi_{n-1}}(F), \\ f(bm) = \chi(b) \delta_{B(G_{2n+1})}^{\frac{1}{2}} f(m) \ \forall b \in B(L_{n-1}^{\Psi_{n-1}}(F)), \ m \in L_{n-1}^{\Psi_{n-1}}(F) \end{array} \right\}.$$

Set  $V = c - \text{ind}_{Q_{n-1}^{\Psi_{n-1}}(F) \cap w_0^{-1}P_{2n}(F)w_0}^{Q_{n-1}^{\Psi_{n-1}}(F)} \hat{\mu} \delta_{P_{2n}}^{\frac{1}{2}} \circ \text{Ad}(w_0)$ . For  $f \in V$ , let

$$W(f)(q) = \int_{N_{n-1}(F) \cap w_0^{-1}\overline{U_{\max}(F)}w_0} f(uq) \bar{\Psi}_{n-1}(u) du.$$

**Lemma 10.0.6.** *The function  $W$  maps  $V$  into  $\mathcal{W}$ .*

*Proof.* Note that an element of  $V$  is left-invariant by  $N_{n-1} \cap w_0^{-1}U_{\max}(F)w_0$ , and that  $\Psi_{n-1}|_{N_{n-1} \cap w_0^{-1}U_{\max}(F)w_0}$  is trivial. Given this, it easily follows that for  $f \in V$ ,  $W(f)(uq) = \Psi_{n-1}(u)W(f)(q)$  for all  $u \in N_{n-1}(F)$  and  $q \in Q_{n-1}(F)$ . Further,  $w_0$  conjugates  $B(G_{2n+1})$  into  $P_{2n}$ , and so an element of  $V$  is left- $B(G_{2n+1})$ -equivariant with respect to a certain quasicharacter. The claim is then that the product of this character with the Jacobian of  $\text{Ad}(b)$ ,  $b \in B(G_{2n+1})(F)$ , acting on  $N_{n-1}(F) \cap w_0^{-1}\overline{U_{\max}(F)}w_0$  is  $\chi \delta_{B(G_{2n+1})}^{\frac{1}{2}}$ , which is a straightforward calculation.  $\square$

Let us denote by  $V(N_{n-1}, \Psi_{n-1})$  the kernel of the linear map  $V \rightarrow \mathcal{J}_{N_{n-1}, \Psi_{n-1}}(V)$ .

It is easy to show that  $V(N_{n-1}, \Psi_{n-1})$  is contained in the kernel of  $W$ . In the next lemma, we show that in fact, they are equal. Restriction from  $Q_{n-1}^{\Psi_{n-1}}(F)$  to  $L_{n-1}^{\Psi_{n-1}}(F)$  is clearly an isomorphism  $\mathcal{W} \rightarrow \text{Ind}_{B(G_{2n+1})(F)}^{G_{2n+1}(F)} \chi$ .  $\square$

**Lemma 10.0.7.** *With notation as in the previous proposition, we have  $\text{Ker}(W) \subset V(N_{n-1}, \Psi_{n-1})$ .*



*Proof.* For this proof, we denote the Borel of  $L_{n-1}^{\Psi_{n-1}}$  by  $B$ . Also, let  $N^{w_0} = N_{n-1} \cap w_0^{-1} P_{2n} w_0$ , and  $N_{w_0} = N_{n-1} \cap w_0^{-1} \overline{U_{\max}} w_0$ ,

We consider a smooth function  $f : Q_{n-1}^{\Psi_{n-1}}(F) \rightarrow \mathbb{C}$  which is compactly supported modulo  $Q_{n-1}^{\Psi_{n-1}}(F) \cap w_0^{-1} P_{2n}(F) w_0$ , and satisfies

$$f(bm) = \chi \delta_B^{\frac{1}{2}}(b) f(m) \quad \forall b \in B(F),$$

and

$$f(uq) = f(q) \quad \forall u \in N^{w_0}(F) \text{ and } q \in Q_{n-1}^{\Psi_{n-1}}(F).$$

We assume that

$$\int_{N_{w_0}(F)} f(uq) \bar{\Psi}_{n-1}(u) du = 0,$$

for all  $q \in Q_{n-1}^{\Psi_{n-1}}(F)$ . What must be shown is that there is a compact subset  $C$  of  $N_{n-1}(F)$  such that

$$\int_C f(gu) \bar{\Psi}_{n-1}(u) du = 0,$$

for all  $q \in Q_{n-1}^{\Psi_{n-1}}(F)$ .

Consider first  $m \in L_{n-1}^{\Psi_{n-1}}(\mathfrak{o})$ . Let  $\mathfrak{p}$  denote the unique maximal ideal in  $\mathfrak{o}$ . If  $U$  is a unipotent subgroup and  $M$  an integer, we define

$$U(\mathfrak{p}^M) = \{u \in U(F) : u_{ij} \in \mathfrak{p}^M \forall i \neq j\}.$$

Observe that for each  $M \in \mathbb{N}$ ,  $N_{n-1}(\mathfrak{p}^M)$  is a subgroup of  $N_{n-1}(F)$  which is preserved by conjugation by elements of  $L_{n-1}^{\Psi_{n-1}}(\mathfrak{o})$ . We may choose  $M$  sufficiently large that  $\text{supp}(f) \subset N^{w_0}(F) N_{w_0}(\mathfrak{p}^{-M}) L_{n-1}^{\Psi_{n-1}}(F)$ . Then we prove the desired assertion with  $C = N_{n-1}(\mathfrak{p}^{-M})$ . Indeed, for  $m \in L_{n-1}^{\Psi_{n-1}}(\mathfrak{o})$ , we have

$$\int_{N_{n-1}(\mathfrak{p}^{-M})} f(mu) \bar{\Psi}_{n-1}(u) du = \int_{N_{n-1}(\mathfrak{p}^{-M})} f(um) \bar{\Psi}_{n-1}(u) du,$$

because  $Ad(m)$  preserves the subgroup  $N_{n-1}(\mathfrak{p}^{-M})$ , and has Jacobian 1. Let  $c = \text{Vol}(N^{w_0}(\mathfrak{p}^{-M}))$ , which is finite. Then by  $N^{w_0}$ -invariance of  $f$ , the above equals

$$= c \int_{N_{w_0}(\mathfrak{p}^{-M})} f(um) \bar{\Psi}_{n-1}(u) du.$$

This, in turn, is equal to

$$= c \int_{N_{w_0}(F)} f(um) \bar{\Psi}_{n-1}(u) du,$$

since none of the points we have added to the domain of integration are in the support of  $f$ , and this last integral is equal to zero by hypothesis.

Next, suppose  $q = u_1 m$  with  $u_1 \in N_{n-1}(F)$  and  $m \in L_{n-1}^{\Psi_{n-1}}(\mathfrak{o})$ . If  $u_1 \in N_{n-1}(F) - N_{n-1}(\mathfrak{p}^{-M})$  then  $qu$  is not in the support of  $f$  for any  $u \in N_{n-1}(\mathfrak{p}^{-M})$ . On the other hand, if  $u_1 \in N_{n-1}(\mathfrak{p}^{-M})$ , then

$$\begin{aligned} \int_{N_{n-1}(\mathfrak{p}^{-M})} f(u_1 m u) \bar{\Psi}_{n-1}(u) du &= \int_{N_{n-1}(\mathfrak{p}^{-M})} f(u_1 u m) \bar{\Psi}_{n-1}(u) du \\ &= \Psi_{n-1}(u_1) \int_{N_{n-1}(\mathfrak{p}^{-M})} f(um) \bar{\Psi}_{n-1}(u) du, \end{aligned}$$

and now we continue as in the case  $u_1 = 1$ .

The result for general  $q$  now follows from the left-equivariance properties of  $f$  and (10.0.2).  $\square$

## 11. APPENDIX II: IDENTITIES OF UNIPOTENT PERIODS

### 11.1. A lemma regarding the projection, and a remark.

**Lemma 11.1.1.** *The action of  $G_m$  on itself by conjugation factors through  $\text{pr}$ .*

*Proof.* One has only to check that the kernel of  $\text{pr}$  is in the center of  $G_m$ . When we regard  $G_m$  as a quotient of  $\text{Spin}_m \times \text{GL}_1$ , the kernel of  $\text{pr}$  is precisely the image of the  $\text{GL}_1$  factor in the quotient.  $\square$

**Corollary 11.1.2.** *Let  $u$  be a unipotent element of  $G_m(\mathbb{A})$  and  $g$  any element of  $G_m(\mathbb{A})$ . Then  $\text{pr}(gug^{-1})$  is a unipotent element of  $\text{SO}_m(\mathbb{A})$  and  $gug^{-1}$  is the unique unipotent element of its preimage in  $G_m(\mathbb{A})$ .*

**Remark 11.1.3.** *Recall that the projection  $\text{pr} : \text{GSpin}_m \rightarrow \text{SO}_m$  induces an isomorphism between the unipotent subvarieties of the two groups. Thus, the unipotent periods of  $\text{GSpin}_m(\mathbb{A})$  and  $\text{SO}_m(\mathbb{A})$  may be identified. It follows from corollary 11.1.2 that any identity or relationship of unipotent periods which is proved using only conjugation and swapping extends to  $\text{GSpin}_m(\mathbb{A})$ . The bulk of this appendix may be viewed as a painstaking check that nearly all the key identities in the descent construction for special orthogonal groups may be proved using only conjugation and swapping.*

**11.2. Relations among Unipotent Periods used in Theorem 9.2.1.** Before we proceed with the proofs it will be convenient to formulate the statements in a slightly different way, making use of the involution  $\dagger$ , introduced in section 6.1.

In section 5.1, we introduced the space  $\mathcal{U}$  of unipotent periods attached to a reductive group  $G(F)$ , as well as an action of  $G(F)$  on  $\mathcal{U}$  by conjugation. In the special case  $G = G_{4n}$ , it is convenient to allow ourselves to conjugate our unipotent periods by elements of the slightly larger group  $\text{Pin}_{4n}$ . We may allow the involution  $\dagger$  to act on unipotent periods by  $f^{\dagger(U, \psi_U)}(g) = f^{(U, \psi_U)}(\dagger g)$ . Denoting the action of  $\text{Pin}_{4n}(F)$  on  $\mathcal{U}$  by  $\gamma \cdot (U, \psi_U)$ , we have

$$\gamma \cdot (U, \psi_U) \sim \begin{cases} (U, \psi_U) & \text{when } \det \text{pr } \gamma = 1, \\ \dagger(U, \psi_U) & \text{when } \det \text{pr } \gamma = -1. \end{cases}$$

Observe that in general  $\dagger(U, \psi_U)$  is *not* equivalent to  $(U, \psi_U)$ . For example, it is not difficult to verify that  $\dagger(U_{\max}, \psi_{LW}) \in \mathcal{U}^{\perp}(\mathcal{E}_{-1}(\tau, \omega))$ .

We shall let  $(U_1, \psi_1)$  and  $(U_3, \psi_3)$  be defined as in the proof of 9.2.1. We also keep the definition of the group  $U_2$ . However, we now define the character  $\psi_2$  by the formula

$$\psi_2(u) = \psi_0(u_{13} + \cdots + u_{2n-1, 2n+1}),$$

regardless of the parity of  $n$ . (This agrees with the previous definition if  $n$  is even; if  $n$  is odd they differ by an application of  $\dagger$ .)

**Lemma 11.2.1.** *Let  $(U_1, \psi_1)$  be defined as in Theorem 9.2.1, and  $(U_2, \psi_2)$  defined as just above. Then  $(U_1, \psi_1)|(U_2, \psi_2)$  and  $(U_1, \psi_1)|\dagger(U_2, \psi_2)$ .*

*Proof.* We define some additional unipotent periods which appear at intermediate stages in the argument. Let  $U_4$  be the subgroup defined by  $u_{n-1, j} = 0$  for  $j = n$  to  $2n - 2$  and  $u_{2n-1, 2n} = u_{2n-1, 2n+1}$ . We define a character  $\psi_4$  of  $U_4$  by the same formula as  $\psi_1$ . Then  $(U_1, \psi_1)$  may be swapped for  $(U_4, \psi_4)$ . (See definition 5.1.3.)

Now, for each  $k$  from 1 to  $n$ , define  $(U_5^{(k)}, \psi_5^{(k)})$  as follows. First, for each  $k$ , the group  $U_5^{(k)}$  is contained in the subgroup of  $U_{\max}$  defined by,  $u_{2n-1, 2n} = u_{2n-1, 2n+1}$ . In addition,  $u_{n+k-2, j} = 0$  for

$j < 2n - 1$ , and  $u_{i,i+1} = 0$  if  $n - k \leq i < n + k$  and  $i \equiv n - k \pmod{2}$ , and  $\psi_5^{(k)}(u)$  equals

$$\psi_0 \left( \sum_{i=1}^{n-k-1} u_{i,i+1} + \sum_{i=n-k}^{n+k-3} u_{i,i+2} + u_{n+k-2,2n} + u_{n+k-2,2n+1} + \sum_{i=n+k-1}^{2n-1} u_{i,i+1} \right).$$

(Note that one or more of the sums here may be empty.)

Next, let  $U_6^{(k)}$  be the subgroup of  $U_{\max}$  defined by the conditions  $u_{2n-1,2n} = u_{2n-1,2n+1}$ ,  $u_{n+k-2,j} = 0$  for  $j < 2n - 1$ , and  $u_{i,i+1} = 0$  if  $n - k \leq i < n + k - 2$  and  $i \equiv n - k + 1 \pmod{2}$ . The same formula which defines  $\psi_5^{(k)}$  also defines a character of  $U_6^{(k)}$ . We denote this character by  $\psi_6^{(k)}$ .

We make the following observations:

- $(U_5^{(1)}, \psi_5^{(1)})$  is precisely  $(U_4, \psi_4)$ .
- For each  $k$ ,  $(U_5^{(k)}, \psi_5^{(k)})$  is conjugate to  $(U_6^{(k+1)}, \psi_6^{(k+1)})$ . The conjugation is accomplished by any preimage of the permutation matrix which transposes  $i$  and  $i + 1$  for  $n - k \leq i < n + k$  and  $i \equiv n - k \pmod{2}$ .
- $(U_6^{(k)}, \psi_6^{(k)})$  may be swapped for  $(U_5^{(k)}, \psi_5^{(k)})$ .

Thus  $(U_4, \psi_4) \sim (U_5^{(n)}, \psi_5^{(n)})$ .

Now, let  $\psi'_2$  be the character of  $U_2$  which is defined by

$$\psi'_2(u) = \psi_0(u_{1,3} + \cdots + u_{2n-2,2n} - u_{2n-2,2n+1} + u_{2n-1,2n+1}).$$

Then  $U_5^{(n)}$  is the subgroup of  $U_2$  defined by  $u_{2n-1,2n} = u_{2n-1,2n+1}$  and  $\psi_5^{(n)}$  is the restriction of  $\psi'_2$  to this group. Thus  $(U_5^{(n)}, \psi_5^{(n)}) | (U_2, \psi'_2)$ . (It is because of this step that  $(U_1, \psi_1) \not\sim (U_2, \psi_2)$ .)

Finally,  $(U_2, \psi_2)$  and  $(U_2, \psi'_2)$  are conjugate by the unipotent element which projects to  $I_{4n} - \sum_{i=2}^n e'_{2i-1,2i-2}$

To obtain  $\dagger(U_2, \psi_2)$ , we use

$$\psi''_2(u) := \psi_0(u_{1,3} + \cdots + u_{2n-2,2n} - u_{2n-2,2n+1} + u_{2n-1,2n})$$

instead of  $\psi'_2$ . □

**Lemma 11.2.2.** *Let  $(U_3, \psi_3)$  be defined as in Theorem 9.2.1, and let  $(U_2, \psi_2)$  be defined as in the previous lemma. Then*

$$(U_3, \psi_3) \in \langle \dagger^n(U_2, \psi_2), \{(N_\ell, \vartheta) : n \leq \ell < 2n \text{ and } \vartheta \text{ in general position.}\} \rangle.$$

Here  $\dagger^n$  indicates that we apply  $\dagger$  a total of  $n$  times, with the effect being  $\dagger$  if  $n$  is odd and trivial if  $n$  is even.

*Proof.* To prove this assertion we introduce some additional unipotent periods. For  $k = 1$  to  $2n - 1$  let  $U_7^{(k)}$  denote the subgroup of  $U_{\max}$  defined by  $u_{i,i+1} = 0$  for  $i > k$  and  $i \equiv k + 1 \pmod{2}$ . We use two characters of this group:

$$\begin{aligned} \tilde{\psi}_7^{(k)} &= \psi_0 \left( \sum_{1 \leq i \leq k-1} u_{i,i+1} + \sum_{k \leq i \leq 2n-1} u_{i,i+2} \right), \\ \psi_7^{(k)} &= \psi_0 \left( \sum_{1 \leq i \leq k} u_{i,i+1} + \sum_{k+1 \leq i \leq 2n-1} u_{i,i+2} \right). \end{aligned}$$

Then  $(U_7, \psi_7^{(k)})$  is conjugate to  $(U_7, \tilde{\psi}_7^{(k)})$  by any preimage of the permutation matrix which transposes  $i$  and  $i + 1$  for  $k < i < 4n - k$  and  $i \equiv k + 1 \pmod{2}$ . This matrix has determinant  $-1$  iff  $k$  is odd.

If  $k$  is odd then  $(U_7^{(k)}, \psi_7^{(k)})$  may be swapped for  $(U_7^{(k+1)}, \tilde{\psi}_7^{(k+1)})$ , while if  $k$  is even, it may be swapped for  $(U_8^{(k+1)}, \tilde{\psi}_8^{(k+1)})$ , where  $U_8^{(k+1)}$  is the subgroup of  $U_7^{(k+1)}$  defined by  $u_{2n-1,2n} = 0$ , and  $\tilde{\psi}_8^{(k+1)}$  is the restriction of  $\tilde{\psi}_7^{(k+1)}$  to this group.

Now, for  $a \in F^\times$  define a character  $\tilde{\psi}_7^{(k+1,a)}$  of  $U_7^{(k+1)}$  by

$$\tilde{\psi}_7^{(k+1,a)} = \psi_0(u_{1,2} + \cdots + u_{k-1,k} + u_{k,k+2} + \cdots + u_{2n-1,2n+1} + au_{2n-1,2n}).$$

Then a Fourier expansion along  $U_{2n-1,2n}$  shows that

$$(U_8^{(k+1)}, \tilde{\psi}_8^{(k+1)}) \in \langle (U_7^{(k+1)}, \tilde{\psi}_7^{(k+1)}), \{(U_7^{(k+1)}, \tilde{\psi}_7^{(k+1,a)}) : a \in F^\times\} \rangle.$$

Here  $U_{ij} = \{u \in U_{\max} : u_{k,\ell} = 0, \forall (k,\ell) \neq (i,j)\}$ .

In Lemma 11.2.3 below we prove that for  $k$  even and  $a \in F^\times$ ,

$$(N_{n+\frac{k}{2}}, \Psi_{n+\frac{k}{2},a})|(U_7^{(k+1)}, \tilde{\psi}_7^{(k+1,a)}),$$

where

$$\Psi_{\ell,a}(u) = \psi_0(u_{1,2} + \cdots + u_{\ell-1,\ell} + au_{\ell,2n} + u_{\ell,2n+1}).$$

The present lemma then follows from the following observations:

- $(U_7^{(1)}, \tilde{\psi}_7^{(1)}) = (U_2, \psi_2)$ , (with  $\psi_2$  defined as at the beginning of this section).
- $(U_7^{(2n-1)}, \psi_7^{(2n-1)}) = (U_3, \psi_3)$
- If one applies  $\dagger$  to both sides of a relation among unipotent periods, it remains valid.
- The character  $\Psi_{n+\frac{k}{2},a}$  of  $N_{n+\frac{k}{2}}$  is in general position. (Cf. remarks 9.1.2)
- The set  $\{(N_\ell, \vartheta) : n \leq \ell < 2n \text{ and } \vartheta \text{ in general position}\}$  is stable under  $\dagger$ .
- The number of times we conjugate by the preimage of an element of determinant minus 1 in passing from  $(U_7^{(k)}, \tilde{\psi}_7^{(k)})$  back to  $(U_7^{(k)}, \psi_7^{(k)})$  is precisely  $n$ .

□

**Lemma 11.2.3.** *Let  $(N_{n+\frac{k}{2}}, \Psi_{n+\frac{k}{2},a})$  and  $(U_7^{(k+1)}, \tilde{\psi}_7^{(k+1,a)})$  be defined as in the previous lemma. Then*

$$(N_{n+\frac{k}{2}}, \Psi_{n+\frac{k}{2},a})|(U_7^{(k+1)}, \tilde{\psi}_7^{(k+1,a)}).$$

*Proof.* We regard  $a$  as fixed for the duration of this argument, and omit it from the notation. We need still more unipotent periods. Specifically, for each  $k, \ell$  define  $U_9^{(k,\ell)}$  to be the subgroup of  $U_{\max}$  defined by requiring that  $u_{ij} = 0$  under any of the following conditions:

$$k < i \leq k + 2\ell, \ i \equiv k + 1 \pmod{2} \text{ and } j = i + 1$$

$$i > k + 2\ell$$

$$i = k + 2\ell - 1, \text{ and } j \neq 4n + 1 - k - 2\ell,$$

$$i = k + 2\ell \text{ and } j < 2n.$$

The formula

$$\psi_0(u_{1,2} + \cdots + u_{k-1,k} + u_{k,k+2} + u_{k+1,k+3} + \cdots + u_{k+2\ell-2,k+2\ell} + au_{k+2\ell,2n} + u_{k+2\ell,2n+1})$$

defines a character of this group which we denote  $\psi_9^{(k,\ell)}(u)$ . Also, let  $U_{10}^{(k,\ell)}$  denote the subgroup of  $U_{\max}$  defined by requiring that  $u_{ij} = 0$  under any of the following conditions:

$$k < i \leq k + 2\ell, \ i \equiv k + 1 \pmod{2} \text{ and } j = i + 1$$

$$i > k + 2\ell - 1$$

$$i = k + 2\ell - 1 \text{ and } j > 2n, 2n + 1.$$

The formula

$$\psi_0(u_{1,2} + \cdots + u_{k,k+1} + u_{k+1,k+3} + \cdots + u_{k+2\ell-2,k_{k+2\ell}} + au_{k+2\ell-1,2n} + u_{k+2\ell-1,2n+1})$$

defines a character of this group which we denote  $\psi_{10}^{(k,\ell)}(u)$ . The period  $(U_9, \psi_9^{(k,\ell)})$  is conjugate to  $(U_{10}, \psi_{10}^{(k,\ell)})$ .

Let  $U_{11}^{(k,\ell)}$  denote the subgroup of  $U_{\max}$  defined by requiring that  $u_{ij} = 0$  under any of the following conditions:

$$\begin{aligned} k < i \leq k + 2\ell, \quad i \equiv k \pmod{2} \text{ and } j = i + 1 \\ i > k + 2\ell - 1 \\ i = k + 2\ell - 1 \text{ and } j > 2n, 2n + 1. \end{aligned}$$

Then  $(U_{10}, \psi_{10}^{(k,\ell)})$  may be swapped for  $(U_{11}, \psi_{11}^{(k,\ell)})$ , where  $\psi_{11}^{(k,\ell)}$  is defined by the same formula as  $\psi_{10}^{(k,\ell)}$ .

Also,  $(U_{11}, \psi_{11}^{(k,\ell)})$ , is clearly divisible by  $(U_9, \psi_9^{(k+1,\ell-1)})$ : to pass from the former to the latter one simply drops the integration over  $u_{k+2\ell-2,j}$ , for  $j \neq 4n - k - 2\ell + 2$ .

To complete the argument: for  $k$  even the period  $(U_9^{(k+1,n-\frac{k}{2}-1)}, \psi_9^{(k+1,n-\frac{k}{2}-1)})$  divides the period  $(U_7^{(k+1)}, \tilde{\psi}_7^{(k+1,a)})$ . Indeed the only difference between the two is that in the former, we omit integration over  $u_{2n-2,2n}$ .

It follows that  $(U_9^{(k+1,n-\frac{k}{2}-1)}, \psi_9^{(k+1,n-\frac{k}{2}-1)})$  is divisible by  $(U_{10}^{n+\frac{k}{2}-1,1}, \psi_{10}^{n+\frac{k}{2}-1,1})$ . Finally, every extension of  $\psi_{10}^{n+\frac{k}{2}-1,1}$  to a character of  $N_{n+\frac{k}{2}}$  is in the same orbit as  $\Psi_{n+\frac{k}{2},a}$ . (See Remarks 9.1.2.) Hence

$$(U_{10}^{n+\frac{k}{2}-1,1}, \psi_{10}^{n+\frac{k}{2}-1,1}) \sim (N_{n+\frac{k}{2}}, \Psi_{n+\frac{k}{2},a}).$$

The result follows.  $\square$

**Lemma 11.2.4.** *As in Theorem 9.2.1, let  $V_i$  denote the unipotent radical of the standard parabolic of  $G_{4n}$  having Levi isomorphic to  $GL_i \times G_{4n-2i}$  (for  $1 \leq i \leq 2n-2$ ). Let  $V_i^{4n-2m-1}$  denote the unipotent radical of the standard maximal parabolic of  $G_{2n+1}$  (embedded into  $G_{4n}$  as  $L_{n-1}^{\Psi_{n-1}}$ ) having Levi isomorphic to  $GL_i \times G_{2n-2i+1}$  (for  $1 \leq i \leq n$ ). Let  $(N_\ell, \Psi_\ell)$  be the period used to define the descent, as usual, and let  $(N_\ell, \Psi_\ell)^{(4n-2k)}$  denote the analogue for  $G_{4n-2k}$ , embedded into  $G_{4n}$  inside the Levi of a maximal parabolic.*

*Then,  $(V_k^{2n+1}, \mathbf{1}) \circ (N_{n-1}, \Psi_{n-1})$  is an element of*

$$\langle (N_{n+k-1}, \Psi_{n+k-1}), \{(N_{n+j-1}, \Psi_{n+j-1})^{(4n-2k+2j)} \circ (V_{k-j}, \mathbf{1}) : 1 \leq j < k\} \rangle.$$

*Proof.* In this proof, we shall not need to refer to any of the unipotent periods defined previously. On the other hand we will need to consider several new unipotent periods. For convenience, we start the numbering over from one.

Thus, let  $(U_1, \psi_1) = (V_k^{2n+1}, \mathbf{1}) \circ (N_{n-1}, \Psi_{n-1})$ . To describe this group and character in detail,  $U_1$  is the subgroup defined by  $u_{ij} = 0$  if  $n-1 < i \leq n-1+k < j$ , or  $n-1+k < i$  and  $u_{i,2n} = u_{i,2n+1}$  if  $n-1 < i \leq n-1+k$ , and  $\psi_1$  is given by

$$\psi_1(u) = \psi_0(u_{1,2} + \cdots + u_{n-2,n-1} + u_{n-1,2n} - u_{n-1,2n+1}).$$

Next, let  $U_2$  denote the subgroup of  $U_1$  defined by the additional conditions  $u_{ij} = 0$  for  $1 \leq i \leq n-1 < j \leq n-1+k$ . Let  $\psi_2$  denote the restriction of  $\psi_1$  to this subgroup.

Next, let  $U_3$  denote the subgroup defined by  $u_{ij} = 0$  for  $i \leq k, j \leq n-1+k$ , and  $i > n-1+k$ , and  $u_{i,2n} = u_{i,2n+1}$  for  $i \leq k$ . Let

$$\psi_3(u) = \psi_0(u_{k+1,k+2} + \cdots + u_{k+n-2,k+n-1} + u_{k+n-1,2n} - u_{k+n-1,2n+1}).$$

Then  $(U_2, \psi_2)$  is conjugate to  $(U_3, \psi_3)$ , by any element of  $G_{4n}(F)$  which projects to

$$\begin{pmatrix} & I_k & & \\ I_{n-1} & & & \\ & I_{4n-2m-2k} & & \\ & & I_{n-1} & \\ & & & I_k \end{pmatrix}$$

(cf. subsection 11.1).

Finally, let  $U_4 \supset U_3$  denote the subgroup of  $U_{\max}$  given by  $u_{ij} = 0$  if  $j \leq k+1$ , or  $i \geq n+k$ . Then take  $\psi_4$  defined by the same formula as  $\psi_3$ .

Certainly  $(U_2, \psi_2)|(U_1, \psi_1)$ , and  $(U_2, \psi_2) \sim (U_3, \psi_3)$ . In Lemma 11.2.5 we prove that  $(U_3, \psi_3) \sim (U_4, \psi_4)$ . It follows that  $(U_4, \psi_4)|(U_1, \psi_1)$ . In fact, one may prove by an argument similar to the proof of Lemma 11.2.5 that in fact  $(U_2, \psi_2) \sim (U_1, \psi_1)$  and hence  $(U_4, \psi_4) \sim (U_1, \psi_1)$ . But this is not needed for our purposes.

Next, let  $U^{(r)}$  denote the subgroup of  $U_{\max}$  defined by  $u_{ij} = 0$  for  $j \leq r$ , or  $i \geq n+k$ . So,  $U_4 = U^{(k+1)}$ , and  $N_{n+k-1} = U^{(1)}$ .

Let  $\psi^{(r)}$  denote the character of  $U^{(r)}$  defined by

$$\psi^{(r)}(u) = \psi_0 \left( \sum_{i=r}^{n-2+k} u_{i,i+1} + u_{n-1+k,2n} + u_{n-1+k,2n+1} \right).$$

Then  $(U_4, \psi_4) = (U^{(k+1)}, \psi^{(k+1)})$ , and  $(N_{n+k-1}, \Psi_{n+k-1}) = (U^{(1)}, \psi^{(1)})$ . It is an easy consequence of Lemma 5.1.1 that

$$(U^{(r)}, \psi^{(r)}) \in \langle (U^{(r-1)}, \psi^{(r-1)}), (N_{n+k-r}, \Psi_{n+k-r})^{(4n-2r+2)} \circ (V_{r-1}, \mathbf{1}) \rangle.$$

The result follows.  $\square$

**Lemma 11.2.5.** *Let  $(U_3, \psi_3)$  and  $(U_4, \psi_4)$  be defined as in the previous lemma. Then  $(U_4, \psi_4) \sim (U_3, \psi_3)$ .*

*Proof.* It's clear that  $(U_3, \psi_3)|(U_4, \psi_4)$ , so we only need to prove that  $(U_4, \psi_4)|(U_3, \psi_3)$ . The proof involves a family of groups defining intermediate stages. For  $\ell$  such that  $1 \leq \ell \leq n-1$  we define  $U_4^{(\ell)}$  to be the subgroup of  $U_4$  defined by the condition that for  $i \leq k$  the coordinate  $u_{ij}$  must be zero for  $j \leq k+\ell$ . Thus  $U_4 = U_4^{(1)} \supset U_4^{(2)} \supset \dots \supset U_4^{(n-1)} \supset U_3$ . For each of these groups we consider the period defined using the restriction of  $\psi_4$ .

We must show that  $(U_4^{(n-1)}, \psi_4)|(U_3, \psi_3)$  and that  $(U_4^{(i)}, \psi_4)|(U_4^{(i-1)}, \psi_4)$ . In each case, all that is involved is an invocation of Lemma 5.1.1. For the first application, what must be checked is that the normalizer of  $U_4(F)$  in  $G(F)$  permutes  $\{\psi'_4 : \psi'_4|_{U_3} = \psi_3\}$  transitively. Let  $y(\underline{r}) = y(r_1, \dots, r_k)$  denote the unipotent element in  $G_{4n}(F)$  which projects to  $I + r_1 e'_{1,2n} + \dots + r_k e'_{k,2n}$ . Then every element of  $U_4^{(m)}$  is uniquely expressible as  $u_3 y(\underline{r})$ , for  $u_3 \in U_3$  and  $\underline{r} \in \mathbb{G}_a^k$ . Hence a map  $\psi'_4$  as above is determined by its composition with  $y$ , which defines a character of  $(F \backslash \mathbb{A})^k$ , and hence is of the form

$$(r_1, \dots, r_k) \mapsto \psi_0(a_1 r_1 + \dots + a_k r_k)$$

for some  $a_1, \dots, a_k \in F$ . Consider the unipotent element  $z(a_1, \dots, a_k)$  of  $G_{4n}$  which projects to  $I + a_1 e'_{k+n-1,1} + \dots + a_k e'_{k+n-1,k}$ . We claim first that it normalizes  $U_4^{(n-1)}$ , and second that  $\psi_4(z(a)y(r)z(a)^{-1}) = \psi_0(a_1 r_1 + \dots + a_k r_k)$ . As noted in 11.1 this may be checked by a matrix multiplication in  $SO_{4n}$ .



The proof that  $(U_4^{(i)}, \psi_4)|(U_4^{(i-1)}, \psi_4)$  is entirely similar, with the role of  $y(\underline{r})$  played by  $y^{(i)}(\underline{r})$  which projects to  $I + r_1 e'_{1,k+i+1} + \cdots + r_k e'_{k,k+i+1}$  and that of  $z(\underline{a})$  played by  $z^{(i)}(\underline{a})$  which projects to  $I + a_1 e'_{k+i,1} + \cdots + a_k e'_{k+i,k}$ .  $\square$

### Part 3. Even case

#### 12. FORMULATION OF THE MAIN RESULT IN THE EVEN CASE

Starting with this section, the “even case” of descent from  $GL_{2n}$  to  $GSpin_{2n}$  will be treated. This material depends on the general matters covered in part 1, but not on the odd case treated in part 2.

Recall the notion of a weak lift which was reviewed in subsection 2.1. For  $\chi$  a nontrivial quadratic character, identify the  $L$  group of  $GSpin_{2n}^\chi$  with  $GO_{2n}(\mathbb{C})$ , and consider the inclusion

$$(12.0.6) \quad r : {}^L G = {}^L(GSpin_{2n}^\chi) \rightarrow GO_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C}) = {}^L GL_{2n} = {}^L H.$$

We are now ready to formulate our main theorem.

**Theorem (MAIN THEOREM: EVEN CASE).** *For  $r \in \mathbb{N}$ , take  $\tau_1, \dots, \tau_r$  to be irreducible unitary automorphic cuspidal representations of  $GL_{2n_1}(\mathbb{A}), \dots, GL_{2n_r}(\mathbb{A})$ , respectively, and let  $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$  be the isobaric sum (see section 2.4). Let  $n = n_1 + \cdots + n_r$ , and assume that  $n \geq 2$ . Let  $\omega$  denote a Hecke character, which is not the square of another Hecke character. Suppose that*

- $\tau_i$  is  $\omega^{-1}$ -orthogonal for each  $i$ , and
- $\tau_i \cong \tau_j \Rightarrow i = j$ .

*For each  $i$ , let  $\chi_i = \omega_{\tau_i}/\omega^{n_i}$  (which is quadratic), and let  $\chi = \prod_{i=1}^r \chi_i$ . Then there exists an irreducible generic cuspidal automorphic representation  $\sigma$  of  $GSpin_{2n}^\chi(\mathbb{A})$  such that*

- $\sigma$  weakly lifts to  $\tau$ , and
- the central character  $\omega_\sigma$  of  $\sigma$  is  $\omega$ .

In fact, a refinement of this theorem with an explicit description of  $\sigma$  is given in theorem 16.3.1, and proved in section 16.

**Remark 12.0.7.** *As a consistency check, we note that the case  $n = 1$  of theorem 12 follows from earlier work of Labesse-Langlands [L-L]. See also [Kaz].*

*Indeed, when  $n = 1$ , the function  $L(s, \tau, \text{sym}^2 \times \omega^{-1})$  has a pole iff  $\chi$  is nontrivial, because  $L(s, \tau, \wedge^2 \times \omega^{-1}) = L(s, \chi)$ . In this case the representation  $\tau$  that we consider is a cuspidal automorphic representation of  $GL(2, \mathbb{A})$ . It is known that in this case  $\tilde{\tau} = \tau \otimes \omega_\tau^{-1}$  (see, e.g., [B], Theorem 3.3.5, p. 305). It follows that our original  $L$ -function on  $\tau$  is, in this case, equivalent to requiring that  $\tau = \tau \otimes \chi$  for some nontrivial quadratic character  $\tau$ . The automorphic representation obtained from the descent construction in this case is simply a character of  $\text{Res}_F^E GL_1(\mathbb{A})$ , where  $E$  is the quadratic extension of  $F$  corresponding to  $\chi$ . Thus, we have recovered proposition 6.5, p. 771 of [L-L]. We thank H. Jacquet for explaining this to us.*

#### 13. NOTATION

**13.1. Siegel parabolic.** In this case, we will construct an Eisenstein series on  $G_{2m+1}$  induced from a standard parabolic  $P = MU$  such that  $M$  is isomorphic to  $GL_m \times GL_1$ . There is a unique such parabolic. We shall refer to this parabolic as the “Siegel.”

**Remark 13.1.1.** • *We can identify the based root datum of the Levi  $M$  with that of  $GL_m \times GL_1$  in such a fashion that  $e_0$  corresponds to  $GL_1$  and does not appear at all in  $GL_m$ . We can then identify  $M$  itself with  $GL_m \times GL_1$  via a particular choice of isomorphism which is compatible with this and with the usual usage of  $e_i, e_i^*$  for characters, cocharacters of the standard torus of  $GL_m$ .*

- Having made this identification, a Levi  $M'$  which is contained in  $M$  will be identified with  $GL_1 \times GL_{m_1} \times \dots GL_{m_k}$ , (for some  $m_1, \dots, m_k \in \mathbb{N}$  that add up to  $m$ ) in the natural way:  $GL_1$  is identified with the  $GL_1$  factor of  $M$ , and then  $GL_{m_1} \times \dots GL_{m_k}$  is identified with the subgroup of  $M$  corresponding to block diagonal elements with the specified block sizes, in the specified order.
- The lattice of rational characters of  $M$  is spanned by the maps  $(g, \alpha) \mapsto \alpha$  and  $(g, \alpha) \mapsto \det g$ . Restriction defines an embedding  $X(M) \rightarrow X(T(G_{2m+1}))$ , which sends these maps to  $e_0$  and  $(e_1 + \dots + e_m)$ , respectively. By abuse of notation, we shall refer to the rational character of  $M$  corresponding to  $e_0$  as  $e_0$  as well.
- The modulus of  $P$  is  $(g, \alpha) \rightarrow \det g^m$ .

### 13.2. Weyl group of $GSpin_{2m+1}$ ; it's action on standard Levis and their representations.

Recall lemma 6.2.1, which establishes an isomorphism between the Weyl groups of  $G_m$  and  $SO_m$ . One easily checks that every element of the Weyl group of  $SO_{2n+1}$  is represented by a matrix of the form  $w = w_0 \det w_0$ , where  $w_0$  is a permutation matrix. We denote the permutation associated to  $w_0$  also by  $w_0$ . The set of permutations  $w_0$  obtained is precisely the set of permutations  $w_0 \in \mathfrak{S}_{2n}$  satisfying,  $w_0(2n+2-i) = 2n+2-w_0(i)$ . It is well known that the Weyl group of  $SO_{2n+1}$  (or  $G_{2n+1}$ ) is isomorphic to  $\mathfrak{S}_n \rtimes \{\pm 1\}^n$ . To fix a concrete isomorphism, we identify  $p \in \mathfrak{S}_n$  with an  $n \times n$  matrix in the usual way, and then with

$$\begin{pmatrix} p & & \\ & 1 & \\ & & {}_t p^{-1} \end{pmatrix} \in SO_{2n}.$$

We identify  $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$  with the permutation  $p$  of  $\{1, \dots, 2n+1\}$  such that

$$p(i) = \begin{cases} i & \text{if } \epsilon_i = 1, \\ 2n+2-i & \text{if } \epsilon_i = -1. \end{cases}$$

We then identify  $(p, \underline{\epsilon}) \in \mathfrak{S}_n \times \{\pm 1\}^n$  (direct product of sets) with  $p \cdot \underline{\epsilon} \in W_{SO_{2n+1}}$ .

With this identification made,

(13.2.1)

$$(p, \underline{\epsilon}) \cdot \begin{pmatrix} t_1 & & & & \\ & \ddots & & & \\ & & t_n & & \\ & & & 1 & \\ & & & & t_n^{-1} \\ & & & & & \ddots \\ & & & & & & t_1^{-1} \end{pmatrix} \cdot (p, \underline{\epsilon})^{-1} = \begin{pmatrix} t_{p^{-1}(1)}^{\epsilon_{p^{-1}(1)}} & & & & \\ & \ddots & & & \\ & & t_{p^{-1}(n)}^{\epsilon_{p^{-1}(n)}} & & \\ & & & 1 & \\ & & & & t_{p^{-1}(n)}^{-\epsilon_{p^{-1}(n)}} \\ & & & & & \ddots \\ & & & & & & t_{p^{-1}(1)}^{-\epsilon_{p^{-1}(1)}} \end{pmatrix}.$$

**Lemma 13.2.2.** *Let  $(p, \underline{\epsilon}) \in \mathfrak{S}_n \rtimes \{\pm 1\}^{n-1}$  be identified with an element of  $W_{SO_{2m}} = W_{G_{2m}}$  as above. Then the action on the character and cocharacter lattices of  $G_{2m}$  is given as follows:*

$$(p, \underline{\epsilon}) \cdot e_i = \begin{cases} e_{p(i)} & i > 0, \epsilon_{p(i)} = 1, \\ -e_{p(i)} & i > 0, \epsilon_{p(i)} = -1, \\ e_0 + \sum_{\epsilon_{p(i)} = -1} e_{p(i)} & i = 0. \end{cases}$$

$$(p, \underline{\epsilon}) \cdot e_i^* = \begin{cases} e_{p(i)}^* & i > 0, \epsilon_{p(i)} = 1, \\ e_0^* - e_{p(i)}^* & i > 0, \epsilon_{p(i)} = -1, \\ e_0^* & i = 0. \end{cases}$$

**Remark 13.2.3.** *Much of this can be deduced from (13.2.1), keeping in mind that  $w \in W_G$  acts on cocharacters by  $(w \cdot \varphi)(t) = w\varphi(t)w^{-1}$  and on characters by  $(w \cdot \chi)(t) = \chi(w^{-1}tw)$ . However, it is more convenient to give a different proof.*

*Proof.* Let  $\alpha_i = e_i - e_{i+1}$ ,  $i = 1$  to  $n - 1$  and  $\alpha_n = e_n$ . Let  $s_i$  denote the elementary reflection in  $W_{G_{2n}}$  corresponding to  $\alpha_i$ . Then it is easily verified that  $s_1, \dots, s_{n-1}$  generate a group isomorphic to  $\mathfrak{S}_n$  which acts on  $\{e_1, \dots, e_n\} \in X(T)$  and  $\{e_1^*, \dots, e_n^*\} \in X^\vee(T)$  by permuting the indices and acts trivially on  $e_0$  and  $e_0^*$ . Also

$$\begin{aligned} s_n \cdot e_i &= \begin{cases} e_i & i \neq n, 0 \\ e_0 + e_n & i = 0 \\ -e_n & i = n \end{cases} \\ s_n \cdot e_i^* &= \begin{cases} e_i^* & i \neq n \\ e_0^* - e_n^* & i = n \end{cases} \end{aligned}$$

If  $\underline{\epsilon} \in \{\pm 1\}^{n-1}$  is such that  $\#\{i : \epsilon_i = -1\} = 1$ , then  $\underline{\epsilon}$  is conjugate to  $s_n$  by an element of the subgroup isomorphic to  $\mathfrak{S}_n$  generated by  $s_1, \dots, s_{n-1}$ . An arbitrary element of  $\{\pm 1\}^{n-1}$  is a product of elements of this form, so one is able to deduce the assertion for general  $(p, \underline{\epsilon})$ .  $\square$

Observe that the  $\mathfrak{S}_n$  factor in the semidirect product is precisely the Weyl group of the Siegel Levi.

In the study of intertwining operators and Eisenstein series (e.g., section 15 below), one encounters a certain subset of the Weyl group associated to a standard Levi,  $M$ . Specifically,

$$W(M) := \left\{ w \in W_{G_{2n+1}} \mid \begin{array}{l} w \text{ is of minimal length in } w \cdot W_M \\ wMw^{-1} \text{ is a standard Levi of } G_{2n+1} \end{array} \right\}.$$

For our purposes, it is enough to consider the case when  $M$  is a subgroup of the Siegel Levi. In this case it is isomorphic to  $GL_{m_1} \times \dots \times GL_{m_r} \times GL_1$  for some integers  $m_1, \dots, m_r$  which add up to  $n$ , and we shall only need to consider the case when  $m_i$  is even for each  $i$ . (This, of course, forces  $n$  to be even as well.)

**Lemma 13.2.4.** *For each  $w \in W(M)$  with  $M$  as above, there exist a permutation  $p \in \mathfrak{S}_r$  and element  $\underline{\epsilon} \in \{\pm 1\}^r$  such that, if  $m \in M = (g, \alpha)$  with  $\alpha \in GL_1$  and*

$$g = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_r \end{pmatrix} \in GL_n,$$

*then*

$$wmw^{-1} = (g', \alpha \cdot \prod_{\epsilon_i = -1} \det g_i) \quad g' = \begin{pmatrix} g'_1 & & \\ & \ddots & \\ & & g'_r \end{pmatrix},$$

*where*

$$g'_i \approx \begin{cases} g_{p^{-1}(i)} & \text{if } \epsilon_{p^{-1}(i)} = 1, \\ {}^t g_{p^{-1}(i)}^{-1} & \text{if } \epsilon_{p^{-1}(i)} = -1. \end{cases}$$

*Here  $\approx$  has been used to denote equality up to an inner automorphism. The map  $(p, \underline{\epsilon}) \mapsto w$  is a bijection between  $W(M)$  and  $\mathfrak{S}_r \times \{\pm 1\}^r$ . (Direct product of sets:  $W(M)$  is not, in general, a group.)*

*Proof.* Since  $wMw^{-1}$  is a standard Levi which does not contain any short roots, it is again contained in the Siegel Levi.

The Levi  $M$  determines an equivalence relation  $\sim$  on the set of indices,  $\{1, \dots, n\}$  defined by the condition that  $i \sim i+1$  iff  $e_i - e_{i+1}$  is a root of  $M$ . When viewed as elements of  $\mathfrak{S}_n \rtimes \{\pm 1\}^{n-1}$ , the elements of  $W(M)$  are those pairs  $(p, \underline{\epsilon})$  such that  $i \sim j \Rightarrow \epsilon_i = \epsilon_j$ , and  $i \sim i+1 \Rightarrow p(i+1) = p(i) + \epsilon_i$ . This gives the identification with  $\mathfrak{S}_r \times \{\pm 1\}^r$ .

It is clear that the precise value of  $g'_i$  is determined only up to conjugacy by an element of the torus (because we do not specify a particular representative for our Weyl group element). By Theorem 16.3.2 of [Spr], it may be discerned, to this level of precision, by looking at the effect of  $w$  on the based root datum of  $M$ . The result now follows from Lemma 13.2.2.  $\square$

**Corollary 13.2.5.** *Let  $w \in W(M)$  be associated to  $(p, \underline{\epsilon}) \in \mathfrak{S}_r \times \{\pm 1\}^r$  as above. Let  $\tau_1, \dots, \tau_r$  be irreducible cuspidal representations of  $GL_{m_1}(\mathbb{A}), \dots, GL_{m_r}(\mathbb{A})$ , respectively, and let  $\omega$  be a Hecke character. Then our identification of  $M$  with  $GL_{m_1} \times \dots \times GL_{m_r} \times GL_1$  determines an identification of  $\bigotimes_{i=1}^r \tau_i \boxtimes \omega$  with a representation of  $M(\mathbb{A})$ . Let  $M' = wMw^{-1}$ . Then  $M'$  is also identified, via 13.1.1 with  $GL_{m_{p^{-1}(1)}} \times \dots \times GL_{m_{p^{-1}(r)}} \times GL_1$ , and we have*

$$\bigotimes_{i=1}^r \tau_i \boxtimes \omega \circ \text{Ad}(w^{-1}) = \bigotimes_{i=1}^r \tau'_i \boxtimes \omega,$$

where

$$\tau'_i = \begin{cases} \tau_{p^{-1}(i)} & \text{if } \epsilon_{p^{-1}(i)} = 1, \\ \tilde{\tau}_{p^{-1}(i)} \otimes \omega & \text{if } \epsilon_{p^{-1}(i)} = -1. \end{cases}$$

*Proof.* The contragredient  $\tilde{\tau}_i$  of  $\tau_i$  may be realized as an action on the same space of functions as  $\tau_i$  via  $g \cdot \varphi(g_1) = \varphi(g_1 t g^{-1})$ . This follows from strong multiplicity one and the analogous statement for local representations, for which see [GK75] page 96, or [BZ1] page 57. Combining this fact with the Lemma, we obtain the Corollary.  $\square$

#### 14. UNRAMIFIED CORRESPONDENCE

**Lemma 14.0.6.** *Suppose that  $\tau \cong \otimes'_v \tau_v$  is an  $\omega^{-1}$ -orthogonal irreducible cuspidal automorphic representation of  $GL_{2n}(\mathbb{A})$ . Let  $v$  be a place such that  $\tau_v$  is unramified. Let  $t_{\tau,v}$  denote the semisimple conjugacy class in  $GL_{2n}(\mathbb{C})$  associated to  $\tau_v$ . Let  $r : GO_{2n}(\mathbb{C}) \rightarrow GL_{2n}(\mathbb{C})$  be the natural inclusion. Then  $t_{\tau,v}$  contains elements of the image of  $r$ .*

*Proof.* For convenience in the application, we take  $GL_{2n}$  to be identified with a subgroup of the Levi of the Siegel parabolic as in section 13.1. Since  $\tau_v$  is both unramified and generic, it is isomorphic to  $\text{Ind}_{B(GL_{2n})(F_v)}^{GL_{2n}(F_v)} \mu$  for some unramified character  $\mu$  of the maximal torus  $T(GL_{2n})(F_v)$  such that this induced representation is irreducible. (See [Car], section 4, [Z] Theorem 8.1, p. 195.) Let  $\mu_i = \mu \circ e_i^*$ .

Since  $\tau \cong \tilde{\tau} \otimes \omega$ , it follows that  $\tau_v \cong \tilde{\tau}_v \otimes \omega_v$  and from this we deduce that  $\{\mu_i : 1 \leq i \leq 2n\}$  and  $\{\mu_i^{-1} \omega : 1 \leq i \leq 2n\}$  are the same set. Hence  $\prod_{i=1}^{2n} \mu_i = \chi \omega^n$ , where  $\chi$  is quadratic.

Now, what we need to prove is the following: if  $S$  is a set of  $2n$  unramified characters of  $F_v$ , such that for each  $i$  there exists  $j$  such that  $\mu_i = \mu_j^{-1} \omega$ , then there is a permutation  $\sigma : \{1, \dots, 2n\} \rightarrow \{1, \dots, 2n\}$  such that  $\mu_{\sigma(i)} = \omega \mu_{2n-\sigma(i)}^{-1}$  for  $i = 1$  to  $n-1$ . This we prove by induction on  $n$ . If  $n = 1$ , there is nothing to be proved.

If  $n > 1$  it is sufficient to show that there exist  $i \neq j$  such that  $\mu_i = \mu_j^{-1} \omega$ . If there exists  $i$  such that  $\mu_i \neq \mu_i^{-1} \omega$  then this is clear. On the other hand, there are exactly two unramified characters  $\mu$  such that  $\mu = \mu^{-1} \omega$ .

Now, suppose that  $\mu_1, \dots, \mu_{2n}$  have been renumbered according to  $\sigma$  as above. Then  $\mu_{n+1}\mu_n = \omega\chi$ . If  $\chi$  is trivial, it follows that  $\mu_i = \omega\mu_{2n-i}^{-1}$  for all  $i$ , and hence that the conjugacy class  $t_{\tau,v}$  contains elements of the maximal torus of  $GO_{2n}(\mathbb{C})$ .

On the other hand, if  $\chi$  is nontrivial, then  $\mu_n \neq \omega\mu_{n+1}^{-1}$ , from which it follows that  $\mu_n^2\mu_{n+1} = \omega$  and  $\mu_{n+1} = \chi\mu_n$ . It follows that  $t_{\tau,v}$  contains a diagonal element which is conjugate, in  $GL_{2n}(\mathbb{C})$ , to an element of the connected component of  $GO_{2n}(\mathbb{C})$  which does not contain the identity.  $\square$

**Corollary 14.0.7.** *Suppose  $\tau = \tau_1 \boxplus \dots \boxplus \tau_r$  with  $\tau_i$  an  $\omega^{-1}$ -orthogonal irreducible cuspidal automorphic representation of  $GL_{2n_i}(\mathbb{A})$ , for each  $i$ . Then the same conclusion holds.*

**Corollary-to-the-Proof 14.0.8.** *Let  $\tau$  be as in corollary 14.0.7, and let  $v$  be a place at which  $\tau$  and  $\omega$  are unramified. Let  $\eta$  be one of the two unramified characters such that  $\eta^2 = \omega_v$ . Let  $\chi_{un}$  denote the unique nontrivial unramified quadratic character of  $F_v^\times$ . Then  $\tau_v \cong \text{Ind}_{B(GL_{2n})(F_v)}^{GL_{2n}(F_v)} \mu$  (normalized induction), for an unramified character  $\mu$  of the torus of  $GL_{2n}(F_v)$  which satisfies either*

$$\mu \circ e_{2n+1-i}^* = \omega_v \cdot (\mu \circ e_i^*)^{-1} \quad \forall i = 1 \text{ to } n,$$

or

$$\mu \circ e_{2n+1-i}^* = \omega_v \cdot (\mu \circ e_i^*)^{-1} \quad \forall i = 1 \text{ to } n-1, \quad \mu \circ e_n^* = \eta, \quad \mu \circ e_{n+1}^* = \chi_{un}\eta.$$

## 15. EISENSTEIN SERIES

The main purpose of this section is to construct, for each integer  $n \geq 2$  and Hecke character  $\omega$ , a map from the set of all isobaric representations  $\tau$  satisfying the hypotheses of theorem 12 into the residual spectrum of  $G_{4n+1}$ . We use the same notation  $\mathcal{E}_{-1}(\tau, \omega)$  for all  $n$ . The construction is given by a multi-residue of an Eisenstein series in several complex variables, induced from the cuspidal representations  $\tau_1, \dots, \tau_r$  used to form  $\tau$ . (Note that by [Ja-Sh3], Theorem 4.4, p.809, this data is recoverable from  $\tau$ .)

Let  $\omega$  be a Hecke character. Let  $\tau_1, \dots, \tau_r$  be a irreducible cuspidal automorphic representations of  $GL_{2n_1}, \dots, GL_{2n_r}$ , respectively.

For each  $i$ , let  $V_{\tau_i}$  denote the space of cuspforms on which  $\tau_i$  acts. Then pointwise multiplication

$$\varphi_1 \otimes \dots \otimes \varphi_r \mapsto \prod_{i=1}^r \varphi_i$$

extends to an isomorphism between the abstract tensor product  $\bigotimes_{i=1}^r V_{\tau_i}$  and the space of all functions

$$\Phi(g_1, \dots, g_r) = \sum_{i=1}^N c_i \prod_{j=1}^r \varphi_{i,j}(g_j) \quad c_i \in \mathbb{C}, \quad \varphi_{i,j} \in V_{\tau_j} \quad \forall i, j.$$

(This is an elementary exercise.) We consider the representation  $\tau_1 \otimes \dots \otimes \tau_r$  of  $GL_{2n_1} \times \dots \times GL_{2n_r}$ , realized on this latter space, which we denote  $V_{\otimes \tau_i}$ .

Let  $n = n_1 + \dots + n_r$ .

We will construct an Eisenstein series on  $G_{4n+1}$  induced from the subgroup  $P = MU$  of the Siegel parabolic such that  $M \cong GL_{2n_1} \times \dots \times GL_{2n_r} \times GL_1$ . Let  $s_1, \dots, s_r$  be a complex variables. Using the identification of  $M$  with  $GL_{2n_1} \times \dots \times GL_{2n_r} \times GL_1$  fixed in section 13.1 above, we define an action of  $M(\mathbb{A})$  on the space of  $\tau_1 \otimes \dots \otimes \tau_r$  by

$$(15.0.9) \quad (g_1, \dots, g_r, \alpha) \cdot \prod_{j=1}^r \varphi_j(h_j) = \left( \prod_{j=1}^r \varphi(h_j g_j) |\det g_j|^{s_j} \right) \omega(\alpha).$$

We denote this representation of  $M(\mathbb{A})$ , by  $(\bigotimes_{i=1}^r \tau_i \otimes |\det \cdot|^{s_i}) \boxtimes \omega$ .

To shorten the notation, we write  $\underline{g} = (g_1, \dots, g_r)$ . Then (15.0.9) may be shortened to

$$\underline{g} \cdot \Phi(\underline{h}) = \Phi(\underline{h} \cdot \underline{g}) \left( \prod_{j=1}^r |\det g_j|^{s_j} \right) \omega(\alpha).$$

We shall also employ the shorthand  $\underline{s} = (s_1, \dots, s_r)$ , and  $\underline{\tau} = (\tau_1, \dots, \tau_r)$ .

For each  $\underline{s}$  we have the induced representation  $\text{Ind}_{P(\mathbb{A})}^{G_{4n+1}(\mathbb{A})} (\bigotimes_{i=1}^r \tau_i \otimes |\det_i|^{s_i}) \boxtimes \omega$ , (normalized induction) of  $G_{4n+1}(\mathbb{A})$ . The standard realization of this representation is action by right translation on the space  $V^{(1)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$  given by

$$\left\{ \tilde{F} : G_{4n+1}(\mathbb{A}) \rightarrow V_{\tau}, \text{ smooth } \left| \tilde{F}((\underline{g}, \alpha)h)(\underline{g}') = \tilde{F}(h)(\underline{g}'\underline{g})\omega(\alpha) \prod_{i=1}^r |\det g_i|^{s_i + n + \sum_{j=i+1}^r n_i - \sum_{j=1}^{i-1} n_i} \right. \right\}.$$

(The factor

$$(15.0.10) \quad \prod_{i=1}^r |\det g_i|^{n + \sum_{j=i+1}^r n_i - \sum_{j=1}^{i-1} n_i}$$

is equal to  $|\delta_P|^{\frac{1}{2}}$ , and makes the induction normalized.) A second useful realization is action by right translation on

$$V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega) = \left\{ f : G_{4n+1}(\mathbb{A}) \rightarrow \mathbb{C}, \left| f(h) = \tilde{F}(h)(id), \tilde{F} \in V^{(1)}(\underline{s}, \underline{\tau}, \omega) \right. \right\}.$$

(Here  $id$  denotes the identity element of  $GL_{2n}(\mathbb{A})$ .)

These representations fit together into a fiber bundle over  $\mathbb{C}^r$ . So a section of this bundle is a function  $f$  defined on  $\mathbb{C}^r$  such that  $f(\underline{s}) \in V^{(i)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$  ( $i = 1$  or  $2$ ) for each  $\underline{s}$ . We shall only require the use of flat,  $K$ -finite sections, which are defined as follows. Take  $f_0 \in V^{(i)}(\underline{0}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$   $K$ -finite, and define  $f(\underline{s})(h)$  by

$$f(\underline{s})(u(\underline{g}, \alpha)k) = f_0(u(\underline{g}, \alpha)k) \prod_{i=1}^r |\det g_i|^{s_i}$$

for  $u \in U(\mathbb{A})$ ,  $\underline{g} \in GL_{2n_1}(\mathbb{A}) \times \dots \times GL_{2n_r}(\mathbb{A})$ ,  $\alpha \in \mathbb{A}^\times$ ,  $k \in K$ . This is well defined. (I.e., although  $g_i$  is not uniquely determined in the decomposition,  $|\det g_i|$  is. Cf. the definition of  $m_P$  on p.7 of [MW1].)

We begin with a flat  $K$  finite section of the bundle of representations realized on the spaces  $V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ .

**Remark 15.0.11.** *Clearly, the function  $f$  is determined by  $f(\underline{s}^*)$  for any choice of base point  $\underline{s}^*$ . In particular, any function of  $f$  may be regarded as a function of  $f_{\underline{s}^*} \in V^{(2)}(\underline{s}^*, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ , for any particular value of  $\underline{s}^*$ . We have exploited this fact with  $\underline{s}^* = 0$  to streamline the definitions. A posteriori it will become clear that the point  $\underline{s}^* = \frac{1}{2} := (\frac{1}{2}, \dots, \frac{1}{2})$  is of particular importance, and we shall then switch to  $\underline{s}^* = \frac{1}{2}$ .*

For such  $f$  the sum

$$E(f)(g)(\underline{s}) := \sum_{\gamma \in P(F) \backslash G(F)} f(\underline{s})(\gamma g)$$

converges for all  $\underline{s}$  such that  $\text{Re}(s_r), \text{Re}(s_i - s_{i+1}), i = 1$  to  $r - 1$  are all sufficiently large. ([MW1], §II.1.5, pp.85-86). It has meromorphic continuation to  $\mathbb{C}^r$  ([MW1] §IV.1.8(a), IV.1.9(c), p.140). These are our Eisenstein series. We collect some of their well-known properties in the following theorem.



**Theorem 15.0.12.** *We have the following:*

(1) *The function*

$$(15.0.13) \quad \prod_{i \neq j} (s_i + s_j - 1) \prod_{i=1}^r (s_i - \frac{1}{2}) E(f)(g)(\underline{s})$$

*is holomorphic at  $s = \frac{1}{2}$ . (More precisely, while  $E(f)(g)$  may have singularities, there is a holomorphic function defined on an open neighborhood of  $\underline{s} = \frac{1}{2}$  which agrees with (15.0.13) on the complement of the hyperplanes  $s_i = \frac{1}{2}$ , and  $s_i + s_j = 1$ .)*

(2) *The function (15.0.13) remains holomorphic (in the same sense) when  $s_i + s_j - 1$  is omitted, provided  $\tau_i \not\cong \omega \otimes \tilde{\tau}_j$ . It remains holomorphic when  $s_i - \frac{1}{2}$  is omitted, provided  $\tau_i$  is not  $\omega^{-1}$ -orthogonal. Furthermore, each of these sufficient conditions is also necessary, in that the holomorphicity conclusion will fail, for some  $f$  and  $g$ , if any of the factors is omitted without the corresponding condition on  $\underline{\tau}$  being satisfied. From this we deduce that if*

$$(15.0.14) \quad \text{the representations } \tau_1, \dots, \tau_r \text{ are all distinct and } \omega^{-1}\text{-orthogonal,}$$

*then the function*

$$(15.0.15) \quad \prod_{i=1}^r (s_i - \frac{1}{2}) E(f)(g)(\underline{s})$$

*is holomorphic at  $s = \frac{1}{2}$  for all  $f, g$  and nonvanishing at  $s = \frac{1}{2}$  for some  $f, g$ .*

(3) *Let us now assume condition (15.0.14) holds, and regard  $f$  as a function of  $f_{\frac{1}{2}} \in V^{(2)}(\frac{1}{2}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ . Let  $E_{-1}(f_{\frac{1}{2}})(g)$  denote the value of the function (15.0.15) at  $\underline{s} = \frac{1}{2}$  (defined by analytic continuation). Then  $E_{-1}(f)$  is an  $L^2$  function for all  $f_{\frac{1}{2}} \in V^{(2)}(\frac{1}{2}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ .*

(4) *The function  $E_{-1}$  is an intertwining operator from  $\text{Ind}_{P(\mathbb{A})}^{G_{4n+1}(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i \otimes |\det_i|^{\frac{1}{2}}) \boxtimes \omega$  into the space of  $L^2$  automorphic forms.*

(5) *If  $\mathcal{E}_{-1}(\tau, \omega)$  is the image of  $E_{-1}$ , and  $\psi_{LW}$  is the character of  $U_{\max}$  given by  $\psi_{LW}(u) = \psi_0(\sum_{i=1}^{2n-1} u_{i,i+1})$ , then  $(U_{\max}, \psi_{LW}) \notin \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$ .*

(6) *The space of functions  $\mathcal{E}_{-1}(\tau, \omega)$  does not depend on the order chosen on the cuspidal representations  $\tau_1, \dots, \tau_r$ . Thus it is well-defined as a function of the isobaric representation  $\tau$ .*

**Remark 15.0.16.** *By induction in stages, the induced representation  $\text{Ind}_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i \otimes |\det_i|^{\frac{1}{2}}) \boxtimes \omega$ , which comes up in part (4) of the theorem can also be written as  $\text{Ind}_{P_{\text{Sie}}(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$ , where  $\tau = \tau_1 \boxplus \dots \boxplus \tau_r$  as before, and  $P_{\text{Sie}}$  is the Siegel parabolic. (Cf. section 2.4.) Here, we also exploit the identification of the Levi  $M_{\text{Sie}}$  of  $P_{\text{Sie}}$  with  $GL_{2n} \times GL_1$  fixed in 13.1.1.*

A detailed proof is given in appendices. First, some preparations are made in section 17. In section 18 theorem 15.0.12 is reduced to a number of lemmas and propositions. These lemmas and propositions are then proved in section 19.

## 16. DESCENT CONSTRUCTION

**16.1. Vanishing of deeper descents and the descent representation.** In this section, we shall make use of remark 15.0.16, and regard  $\mathcal{E}_{-1}(\tau, \omega)$  as affording an automorphic realization of the representation induced from the representation  $\tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$  of the Siegel Levi. Thus we may

dispense with the smaller Levi denoted by  $P$  in the previous section, and in this section we denote the Siegel parabolic more briefly by  $P = MU$ .

Next we describe certain unipotent periods of  $G_{2m}$  which play a key role in the argument. For  $1 \leq \ell < m$ , let  $N_\ell$  be the subgroup of  $U_{\max}$  defined by  $u_{ij} = 0$  for  $i > \ell$ . (Recall that according to the convention above, this refers only to those  $i, j$  with  $i < j \leq m - i$ .) This is the unipotent radical of a standard parabolic  $Q_\ell$  having Levi  $L_\ell$  isomorphic to  $GL_1^\ell \times G_{2m-2\ell}$ .

Let  $\vartheta$  be a character of  $N_\ell$  then we may define

$$DC^\ell(\tau, \omega, \vartheta) = FC^\vartheta \mathcal{E}_{-1}(\tau, \omega).$$

**Theorem 16.1.1.** *Let  $\omega$  be a Hecke character. Let  $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$  be an isobaric sum of  $\omega^{-1}$ -orthogonal irreducible cuspidal automorphic representations  $\tau_1, \dots, \tau_r$ , of  $GL_{2n_1}(\mathbb{A}), \dots, GL_{2n_r}(\mathbb{A})$ , respectively. If  $\ell > n$ , and  $\vartheta$  is in general position, then*

$$DC^\ell(\tau, \omega, \vartheta) = \{0\}.$$

*Proof.* By Theorem 15.0.12, (3) the representation  $\mathcal{E}_{-1}(\tau, \omega)$  decomposes discretely. Let  $\pi \cong \otimes'_v \pi_v$  be one of the irreducible components, and  $p_\pi : \mathcal{E}_{-1}(\tau, \omega) \rightarrow \pi$  the natural projection.

Fix a place  $v_0$  such which  $\tau_{v_0}$  and  $\pi_{v_0}$  are unramified. For any  $\xi^{v_0} \in \otimes'_{v \neq v_0} \text{Ind}_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v$  we define a map

$$i_{\xi^{v_0}} : \text{Ind}_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0} \rightarrow \text{Ind}_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$$

by  $i_{\xi^{v_0}}(\xi_v) = \iota(\xi_{v_0} \otimes \xi^{v_0})$ , where  $\iota$  is an isomorphism of the restricted product  $\otimes'_v \text{Ind}_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v$  with the global induced representation  $\text{Ind}_{P(\mathbb{A})}^{G_{4n}(\mathbb{A})} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega$ . Clearly

$$\mathcal{E}_{-1}(\tau, \omega) = E_{-1} \circ \iota(\otimes'_v \text{Ind}_{P(F_v)}^{G_{4n}(F_v)} \tau_v \otimes |\det|_v^{\frac{1}{2}} \boxtimes \omega_v).$$

For any decomposable vector  $\xi = \xi_{v_0} \otimes \xi^{v_0}$ ,

$$p_\pi \circ E_{-1} \circ \iota(\xi) = p_\pi \circ E_{-1} \circ i_{\xi^{v_0}}(\xi_{v_0}).$$

Thus,  $\pi_{v_0}$  is a quotient of  $\text{Ind}_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0}$ , and hence (since we took  $v_0$  such that  $\pi_{v_0}$  is unramified) it is isomorphic to the unramified constituent  ${}^{un}\text{Ind}_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0}$ .

Denote the isomorphism of  $\pi$  with  $\otimes'_v \pi_v$  by the same symbol  $\iota$ . This time, fix  $\zeta^{v_0} \in \otimes'_{v \neq v_0} \pi_v$ , and define  $i_{\zeta^{v_0}} : {}^{un}\text{Ind}_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0} \rightarrow \pi$ . It follows easily from the definitions that

$$FC^\vartheta \circ i_{\zeta^{v_0}}$$

factors through the Jacquet module  $\mathcal{J}_{N_\ell, \vartheta}({}^{un}\text{Ind}_{P(F_{v_0})}^{G_{4n}(F_{v_0})} \tau_{v_0} \otimes |\det|_{v_0}^{\frac{1}{2}} \boxtimes \omega_{v_0})$ . Propositions 20.0.16 and 20.0.18 below each show that this Jacquet module vanishes at approximately half of all places. Inasmuch as vanishing at a single place would suffice to prove global vanishing, the result follows.  $\square$

A general character of  $N_\ell$  is of the form

$$(16.1.2) \quad \psi_0(c_1 u_{1,2} + \cdots + c_{\ell-1} u_{\ell-1,\ell} + d_1 u_{\ell,\ell+1} + \cdots + d_{4n+1-2\ell} u_{\ell,4n+1-\ell}).$$

As described in section 5, the Levi  $L_\ell$  acts on the space of characters of  $N_\ell(F \backslash \mathbb{A})$ . In order to define embeddings of the various forms of  $G_{2n}$  into  $G_{4n+1}$ , we need to make this more explicit.

First, we fix a specific isomorphism of  $GL_1^\ell \times G_{4n-2\ell+1}$  with  $L_\ell$  as follows. As in section 4.1, let  $e_0, \dots, e_{2n}$  and  $e_0^*, \dots, e_{2n}^*$  denote the  $\mathbb{Z}$ -bases of  $X(T(G_{4n+1}))$  and  $X^\vee(T(G_{4n+1}))$ , respectively. Let  $\hat{e}_0, \dots, \hat{e}_{2n-\ell}$ , and  $\hat{e}_0^*, \dots, \hat{e}_{2n-\ell}^*$  denote the analogues for  $G_{4n-2\ell+1}$ . We identify

$(\alpha_1, \dots, \alpha_\ell, \prod_{i=1}^{2n-\ell} e_i^*(t_i)) \in GL_1^\ell \times T(G_{4n-2\ell+1})$  with  $\prod_{i=1}^\ell e_i^*(\alpha_i) \cdot \prod_{i=1}^{2n-\ell} e_{i+\ell}^*(t_i) \in T(G_{4n+1})$ . In addition, we require that  $g \in G_{4n-2\ell+1}$  be identified with an element of  $G_{4n+1}$  which projects to

$$\begin{pmatrix} I_\ell & & \\ & \text{pr}(g) & \\ & & I_\ell \end{pmatrix} \in SO_{4n+1}.$$

Together, these requirements determine a unique identification.

Let  $\underline{d}$  denote the column vector  ${}^t(d_1, \dots, d_{4n+1-2\ell})$ . Suppose  $\vartheta(u)$  is the character of  $N_\ell$  given by (16.1.2), and, for  $h \in L_\ell$ , let

$$(16.1.3) \quad h \cdot \vartheta(u) = \vartheta(h^{-1}uh) = \psi_0({}^h c_1 u_{1,2} + \dots + {}^h c_{\ell-1} u_{\ell-1,\ell} + {}^h d_1 u_{\ell,\ell+1} + \dots + {}^h d_{4n+1-2\ell} u_{\ell,4n+1-\ell}).$$

This is an action of  $L_\ell$  on the space of characters, and it is easily verified that for  $h$  identified with  $(\alpha_1, \dots, \alpha_\ell, g)$ , with  $\alpha_1, \dots, \alpha_\ell \in GL_1(F)$  and  $g \in G_{4n-2\ell+1}(F)$ , we have

$${}^h c_i = \frac{\alpha_{i+1}}{\alpha_i} \cdot c_i, \quad i = 1, \dots, \ell-1, \quad \text{and} \quad {}^h \underline{d} = \alpha_\ell^{-1} \cdot \text{pr}(g) \cdot \underline{d}.$$

The above discussion amounts to an identification of the action of  $L_\ell(F)$  on the space of characters of  $N_\ell(F \backslash \mathbb{A})$  with a certain rational representation of  $L_\ell$  defined over  $F$ , consisting of the direct sum of  $\ell-1$  one dimensional representations and a  $(4n-2\ell+1)$ -dimensional representation on which the  $G_{4n-2\ell+1}$  factor in  $L_\ell$  acts via its “standard” representation. We may consider this rational representation over any field. Over an algebraically closed field there is an open orbit, which consists of all those elements such that  $c_i \neq 0$  for all  $i$  and  ${}^t \underline{d} J \underline{d} \neq 0$ . Here,  $J$  is defined as in 3.1. Over a general field two such elements are in the same  $F$ -orbit iff the two values of  ${}^t \underline{d} J \underline{d}$  are in the same square class. Thus, this square class is an important invariant of the character  $\vartheta$ .

**Definition 16.1.4.** Let  $\vartheta$  be the character of  $N_\ell(F \backslash \mathbb{A})$  given by

$$\vartheta(u) = \psi_0(c_1 u_{1,2} + \dots + c_{\ell-1} u_{\ell-1,\ell} + d_1 u_{\ell,\ell+1} + \dots + d_{4n+1-2\ell} u_{\ell,4n+1-\ell}).$$

We denote the square class of  ${}^t \underline{d} J \underline{d}$  by  $\text{Inv}(\vartheta)$ . We say that  $\vartheta$  is in general position if  $c_i \neq 0$  for  $1 \leq i \leq \ell-1$  and  $\text{Inv}(\vartheta) \neq 0$ . We denote the square class consisting of the nonzero squares by  $\square$ .

Clearly, a nonzero square class in  $F$  may also be used to determine a quasi-split form of  $G_{2n}$ . Indeed, the natural datum for determining a quasi-split group with  $G$  such that  ${}^L G^0 = GSO_{2n}(\mathbb{C})$  is a homomorphism  $\text{Gal}(\bar{F}/F) \rightarrow \text{Aut}(GSO_{2n}(\mathbb{C}))/\text{Inn}(GSO_{2n}(\mathbb{C}))$ , which has two elements. Such homomorphisms are in one-to-one correspondence with quadratic characters by class field theory, and this has been exploited in defining  $G_{2n}^\chi$  above. But they are also in natural one-to-one correspondence with square classes in  $F^\times$ , and this parametrization will be more convenient for the next part of the discussion.

**Definition 16.1.5.** Let  $\mathbf{a}$  be a square class in  $F^\times$ . Let  $F(\sqrt{\mathbf{a}})$  denote the smallest extension of  $F$  in which the elements of  $\mathbf{a}$  are squares, and for  $a \in F^\times$ , let  $F(\sqrt{a}) = F(\sqrt{\{a\}})$ , where  $\{a\}$  is the square class of  $a$ . Let  $G_{2n}^{\mathbf{a}}$  denote the quasi-split form of  $G_{2n}$  such that the associated map  $\text{Gal}(\bar{F}/F) \rightarrow \text{Aut}(GSO_{2n}(\mathbb{C}))/\text{Inn}(GSO_{2n}(\mathbb{C}))$  factors through  $\text{Gal}(F(\sqrt{\mathbf{a}})/F)$ .

**Remark 16.1.6.** Of course, if  $\mathbf{a} = \square$ , then  $F(\sqrt{\mathbf{a}}) = F$  and  $G_{2n}^{\mathbf{a}}$  is just the split group  $G_{2n}$ .

**Lemma 16.1.7.** (1) If  $\vartheta$  is a character of  $N_\ell$  in general position, then the stabilizer  $L_\ell^\vartheta$  (cf.  $M^\vartheta$  in definition 5.0.2) has two connected components

(2) The identity component  $(L_\ell^\vartheta)^0$  is isomorphic over  $F$  to  $G_{4n-2\ell}^{\text{Inv}(\vartheta)}$ .

*Proof.* Identify  $(\alpha_1, \dots, \alpha_\ell, g) \in GL_1^\ell \times G_{4n-2\ell+1}$  with an element of  $L_\ell$  as above.

The identity component of  $L_\ell^\vartheta$  consists of those  $(\alpha_1, \dots, \alpha_\ell, g)$  such that  $\alpha_i = 1$  for all  $i$  and  $g$  fixes the vector in the standard representation obtained from  $\vartheta$ . The other consists of those such

that  $\alpha_i = -1$  for all  $i$ , and  $g$  maps the vector in the standard representation obtained from  $\vartheta$  to its negative (which is the only scalar multiple of the same length). This proves (1).

We turn to (2). First suppose  $\text{Inv}(\vartheta) = \square$ . It suffices to consider the specific character  $\Psi_\ell$  defined by

$$\Psi_\ell(u) = \psi_0(u_{12} + \cdots + u_{\ell-1,\ell} + u_{\ell,2n+1}).$$

For this character, the column vector  $\underline{d}$  is  $v_1 := {}^t(0, \dots, 0, 1, 0, \dots, 0)$ . It is easily checked that the stabilizer of this point in  $SO_{4n-2\ell+1}$  is isomorphic to the split form of  $SO_{4n-2\ell}$ . In addition, the stabilizer in  $G_{4n-2\ell+1}$  contains a split torus of rank  $2n - \ell + 1$ , and hence is a split group. An element of  $U_{\max}$  fixes  $v_1$ , if and only if it satisfies  $u_{i,2n-\ell+1} = 0$  for  $i = 1$  to  $2n - \ell$ . From this we easily compute the based root datum of the stabilizer of  $v_1$  and find that it is the same as that of  $G_{4n-2\ell}$ .

To complete the proof of (2), let  $a$  be a non-square in  $F^\times$ , and let  $v_a = {}^t(0, \dots, 0, 1, 0, \frac{a}{2}, 0, \dots, 0) \in F^{4n-2\ell+1}$  (nonzero entries in positions  $2n - \ell$  and  $2n - \ell + 2$  only). Let  $\Psi_\ell^a$  be the character of  $N_\ell(F \backslash \mathbb{A})$  corresponding to  $c_i = 1 \forall i$  and  $\underline{d} = v_a$ . The stabilizers of  $\Psi_\ell^a$  and  $\Psi_\ell$  are conjugate over the quadratic extension  $E$  of  $F$  obtained by adjoining a square root of  $a$ . Indeed, let  $\sqrt{a}$  be an element of  $E$  such that  $(\sqrt{a})^2 = a$ . Suppose

$$\text{pr}(h_a) = \begin{pmatrix} \sqrt{a}^{-1} I_{2n-1} & & \\ & M_{\sqrt{a}} & \\ & & \sqrt{a} I_{2n-1} \end{pmatrix}, \quad \text{where} \quad M_{\sqrt{a}} = \begin{pmatrix} -\frac{1}{2\sqrt{a}} & \sqrt{a}^{-1} & \sqrt{a}^{-1} \\ \frac{1}{2} & 0 & 1 \\ \frac{\sqrt{a}}{4} & \frac{\sqrt{a}}{2} & -\frac{\sqrt{a}}{2} \end{pmatrix}.$$

Then  $h_a \cdot \Psi_\ell = \Psi_\ell^a$ . For each  $a$ , fix an element  $h_a$  as above for use throughout.

Clearly  $(L_\ell^{\Psi_\ell^a})^0 = h_a(L_\ell^{\Psi_\ell})^0 h_a^{-1}$ . The image of this group under  $\text{pr}$  is isomorphic over  $F$  to the non-split quasisplit form of  $SO_{4n-2\ell}$  corresponding to the square class of  $a$ . It follows that  $(L_n^{\Psi_\ell^a})^0$  is isomorphic over  $F$  to the non-split quasisplit form of  $G_{4n-2\ell}$  associated to the square class of  $a$ .  $\square$

In the course of the preceding proof, we have seen that it is enough to consider one conveniently chosen representative from each  $F$ -orbit of characters in general position. However, it is generally more convenient to make definitions for general  $a \in F^\times$  than it is to choose representatives for the square classes in  $F^\times$ .

**Definition 16.1.8.** Take  $a \in F^\times$ , and let  $\Psi_\ell^a$  be the character of  $N_\ell$  defined by

$$\Psi_\ell^a(u) = \psi_0(u_{12} + \cdots + u_{\ell-1,\ell} + u_{\ell,2n} + \frac{a}{2}u_{\ell,2n+2}).$$

We also keep the notation

$$\Psi_\ell(u) = \psi_0(u_{12} + \cdots + u_{\ell-1,\ell} + u_{\ell,2n+1}).$$

Then the orbit of  $\Psi_\ell^a$  is determined by the square class of  $a$ . The character  $\Psi_\ell$  is in the same orbit as  $\Psi_\ell^1$ .

Note that for any given square class  $\mathbf{a}$  we have many conjugate embeddings of  $G_{2n}^{\mathbf{a}}$  into  $G_{4n+1}$ : one for each element  $a$  of  $\mathbf{a}$ .

**Definition 16.1.9.** For each element  $a$  of  $F^\times$ , we let  $G_{2n}^a$  denote  $(L_n^{\Psi_n^a})^0$ . It is a subgroup of  $G_{4n+1}$ , which is isomorphic over  $F$  to  $G_{2n}^{\{a\}}$ , where  $\{a\}$  is the square class of  $a$ .

**Lemma 16.1.10.** Assume  $\{a\} \neq \square$ . Then,

- (1) An element  $u$  of  $U_{\max}$  is in  $G_{2n}^a$  iff it satisfies  $u_{ij} = 0$  for  $i \leq n$  or  $i = 2n$ , and  $u_{i,2n} = -\frac{a}{2}u_{i,2n+2}$  for  $n < i < 2n$ . The set of such elements  $u$  is equal to  $h_a(U_{\max} \cap (L_n^{\Psi_n})^0)h_a^{-1}$ , and is a maximal unipotent subgroup of  $G_{2n}^a$ .

- (2) An element  $t = \prod_{i=0}^{2n} e_i^*(t_i)$  of  $T(G_{4n+1})$  is in  $G_{2n}^a$  iff it satisfies  $t_i = 1$  for  $0 < i \leq n$ , and  $i = 2n$ . The set of such  $t$  is a maximal  $F$ -split torus of  $G_{2n}^a$ .
- (3) There is a maximal torus of  $G_{2n}^a$  which contains the above maximal  $F$ -split torus and is contained in the standard Levi of  $G_{4n+1}$  whose unique positive root is the short simple root  $e_n$ . Its set of  $F$  points is equal to

$$\left\{ h_a t h_a^{-1} : t = \prod_{i=1}^{n-1} e_{n+i}^*(t_i) e_{2n}^*(x \cdot \bar{x}^{-1}) e_0^*(\bar{x}), t_1, \dots, t_{n-1} \in F^\times, x \in F(\sqrt{a})^\times \right\},$$

where  $-$  denotes the action of the nontrivial element of  $\text{Gal}(F(\sqrt{a})/F)$ .

If  $\{a\} = \square$ , then (1) remains true, while

$$\left\{ h_a t h_a^{-1} : t = \prod_{i=0}^n e_{n+i}^*(t_i) \right\},$$

is a maximal torus, and is  $F$ -split, since  $h_a$  has entries in  $F$ .

**Remark 16.1.11.** We may write an element of our maximal torus as

$$\left\{ h_a \prod_{i=1}^{n-1} e_{n+i}^*(t_i) \cdot e_{2n}^*((x + y\sqrt{a}) \cdot (x - y\sqrt{a})^{-1}) e_0(x - y\sqrt{a}) h_a^{-1} : t_i \in F, x, y \in F, x^2 - ay^2 \neq 0 \right\},$$

regardless of  $\{a\}$ .

*Proof.* Item (1) is easily checked. (Recall that  $\text{pr}$  is an isomorphism on  $U_{\max}$ .) Similarly, it is easily checked that an element  $t$  of  $T(G_{4n+1})$  stabilizes the specified character iff  $t_1 = \dots = t_n = t_{2n} = \pm 1$ . As noted in the proof of Lemma 16.1.7, if they are all minus 1, then this element is in the other connected component of  $L_n^{\Psi_n^a}$ .

Recall that  $(L_n^{\Psi_n})^0$ , with  $\Psi_n$  as in Definition 16.1.8 is isomorphic to  $G_{2n}$ . There is an “obvious” choice of isomorphism  $\text{inc} : G_{2n} \rightarrow (L_n^{\Psi_n})^0$ , such that

$$\text{inc} \circ \bar{e}_i^* = \begin{cases} e_0^* & i = 0, \\ e_{n+i}^* & 1 \leq i \leq n, \end{cases} \quad \text{and} \quad \text{inc}(u)_{ij} = \begin{cases} 0 & i \leq n, \text{ or } j = 2n + 1, \\ u_{i-n, j-n} & i > n, j < 2n + 1, \\ u_{i-n, j-n-1} & i > n, j > 2n + 1. \end{cases}$$

Here, we have used  $e_i^*$  for elements of the  $\mathbb{Z}$ -basis of the cocharacter lattice of  $G_{4n+1}$  and  $\bar{e}_i^*$  for elements of that of  $G_{2n}$ . It follows from the definitions that conjugation by  $h_a$  is an isomorphism of  $G_{2n}^a$  with  $(L_n^{\Psi_n})^0$ , which is defined over  $F(\sqrt{a})$ . This yields an identification of the maximal  $F$ -split torus of  $G_{2n}^{\{a\}}$  as computed in section 4.1 with the  $F$ -split torus in item (2).

Clearly  $h_a \cdot \text{inc}(T(G_{2n})) \cdot h_a^{-1}$  is a maximal torus of  $G_{2n}^a$ . The fact that an element is of the form specified in item (3) of the present lemma follows from the action of  $\text{Gal}(\bar{F}/F)$  on the lattice of cocharacters computed in section 4.1.  $\square$

Finally, following the works of Ginzburg-Rallis-Soudry we arrive at the *descent construction*.

**Definition 16.1.12.** Let

$$DC_\omega^a(\tau) = FC_\omega^{\Psi_n^a} \mathcal{E}_{-1}(\tau, \omega).$$

It is a space of smooth functions  $G_{2n}^a(F \backslash \mathbb{A}) \rightarrow \mathbb{C}$ , and affords a representation of the group  $G_{2n}^a(\mathbb{A})$  acting by right translation, where we have identified  $G_{2n}^a$  with the identity component of  $L_n^{\Psi_n^a}$ .

**Definition 16.1.13.** We say that a square class  $\mathbf{a}$  in  $F^\times$  and a character  $\chi$  are compatible if they correspond to the same homomorphism from  $\text{Gal}(\bar{F}/F)$  to the group with two elements. We say that an element  $a$  of  $F^\times$  and a character  $\chi$  are compatible if  $\chi$  is compatible with the square class of  $a$ .

## 16.2. Vanishing of incompatible descents.

**Theorem 16.2.1.** *Let  $\omega$  be a Hecke character. Let  $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$  be the isobaric sum of distinct  $\omega^{-1}$ -orthogonal unitary cuspidal automorphic representations of  $GL_{2n_1}(\mathbb{A}), \dots, GL_{2n_r}(\mathbb{A})$ , respectively. For  $i = 1$  to  $r$  let  $\omega_{\tau_i}$  denote the central character of  $\tau_i$  and let  $\chi_i = \omega_{\tau_i}/\omega^{n_i}$ , which is quadratic. Let  $\chi = \prod_{i=1}^r \chi_i$ . Suppose that  $\chi$  and  $a$  are not compatible. Then  $DC_\omega^a(\tau) = \{0\}$ .*

*Proof.* As in Theorem 16.1.1, it suffices to prove the vanishing of the corresponding twisted Jacquet module of  $\text{Ind}_{P(F_v)}^{G_{4n+1}(F_v)} \tau_v \otimes |\det|^{\frac{1}{2}} \boxtimes \omega_v$  at a single unramified place  $v$ . The vanishing follows from Proposition 20.0.16, if there is an unramified place  $v$  such that  $\chi_v$  is trivial and  $a$  is not a square, and from Proposition 20.0.18 if there is an unramified place  $v$  such that  $\chi_v$  is nontrivial and  $a$  is a square. If  $\chi$  and  $a$  are incompatible, then there is at least one unramified place at which one of these cases occurs.  $\square$

## 16.3. Main Result.

**Theorem 16.3.1.** *Let  $\omega$  be a Hecke character. Let  $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$  be the isobaric sum of distinct  $\omega^{-1}$ -orthogonal unitary cuspidal automorphic representations of  $GL_{2n_1}(\mathbb{A}), \dots, GL_{2n_r}(\mathbb{A})$ , respectively. For  $i = 1$  to  $r$  let  $\omega_{\tau_i}$  denote the central character of  $\tau_i$  and let  $\chi_i = \omega_{\tau_i}/\omega^{n_i}$ , which is quadratic. Let  $\chi = \prod_{i=1}^r \chi_i$ . Then*

- (1)  $DC_\omega^a(\tau)$  is nontrivial if and only if  $\chi$  and  $a$  are compatible,
- (2) If  $\chi$  and  $a$  are compatible then the space  $DC_\omega^a(\tau)$  is a nonzero, cuspidal representation of  $G_{2n}^a(\mathbb{A})$ , with central character  $\omega$ . Furthermore, the representation  $DC_\omega^a(\tau)$  supports a nonzero Whittaker integral for the generic character of  $U_{\max}(\mathbb{A}) \cap G_{2n}^a(\mathbb{A})$  given by

$$u \mapsto \psi_0 \left( \sum_{i=1}^{2n-2} u_{i,i+1} + u_{2n-1,2n+2} \right).$$

- (3) If  $\sigma$  is any irreducible automorphic representation contained in  $DC_\omega^a(\tau)$ , then  $\sigma$  lifts weakly to  $\tau$  under the map  $r$ .

**Remark 16.3.2.** *Since  $DC_\omega(\tau)$  is nonzero and cuspidal, there exists at least one irreducible component  $\sigma$ . In the case of special orthogonal groups, one may show ([So1], p. 342, item 4) that the descent module is in the  $\psi$ -generic spectrum for a suitable choice of  $\psi$  (cf. section 3.2). It follows that all of the irreducible components are distinct and globally  $\psi$ -generic. This is done using the Rankin-Selberg integrals of [Gi-PS-R],[So2]. In the odd case, one may also show ([GRS4], Theorem 8, p. 757, or [So1] page 342, item 6) using the results of [Ji-So] that the descent module is irreducible. This does **not** extend to the even case, even for special orthogonal groups, because the construction actually yields a representation of the full stabilizer— which is isomorphic to the full orthogonal group. (Cf. Proposition 7.0.20.)*

**16.4. Proof of the main theorem (Even case).** The statements are proved by combining relationships between unipotent periods and knowledge about  $\mathcal{E}_{-1}(\tau, \omega)$ .

- (1) **Genericity and non-vanishing** For  $a \in F^\times$ , we let  $(U_1^a, \psi_1^a)$  denote the unipotent period obtained by composing the period  $(N_n, \Psi_n^a)$ , used in defining the descent to  $G_{2n}^a$ , (embedded into  $G_{4n+1}$  as the stabilizer of  $\Psi_n^a$ ) with a period which defines a Whittaker integral on this group. Specifically,  $U_1^a$  is the subgroup of the standard maximal unipotent defined by the relations  $u_{i,2n} = -\frac{a}{2}u_{i,2n+2}$  for  $i = n+1$  to  $2n-1$ , as well as  $u_{2n,2n+1} = 0$ , and

$$\psi_1^a(u) = \psi_0(u_{1,2} + \cdots + u_{n-2,n-1} + u_{n-1,2n} + \frac{a}{2}u_{n-1,2n+2} + u_{n,n+1} + \cdots + u_{2n-1,2n}).$$

The definitions of  $U_1^a$  and  $\psi_1^a$  make sense also in the case when  $a = 0$ , although in that case there is no interpretation in terms of a descent. We use this period in that case also.



Next, let  $U_2$  denote the subgroup of the standard maximal unipotent defined by  $u_{2n,2n+1} = 0$ , and  $u_{12} = u_{34} = \cdots = u_{2n-1,2n}$ . For all  $a \in F$ , we may define a character of this group by the formula

$$\psi_2^a(u) = \psi_0 \left( \sum_{i=1}^{2n-2} u_{i,i+2} + u_{2n-1,2n+2} + \frac{a}{2} u_{2n-1,2n} \right).$$

Finally, let  $U_3$  denote the maximal unipotent, and  $\psi_3$  denote

$$\psi_3(u) = \psi_0(u_{1,2} + \cdots + u_{2n-1,2n}).$$

Thus  $(U_3, \psi_3)$  is the composite of the unipotent period defining the constant term along the Siegel parabolic, and one which defines a Whittaker integral on the Levi of this parabolic. By Theorem 15.0.12 (5) this period is *not* in  $\mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$ .

In the appendices, we show

- (a)  $(U_1^a, \psi_1^a) \sim (U_2, \psi_2^a)$ , for all  $a \in F$ , in Lemma 21.0.24,
- (b)  $(U_2, \psi_2^0) \in \langle \{(U_2, \psi_2^a) : a \in F^\times\} \rangle$ , in Lemma 21.0.27, and
- (c)  $(U_3, \psi_3) \in \langle (U_2, \psi_2^0), \{(N_\ell, \vartheta) : n < \ell < 2n \text{ and } \vartheta \text{ in general position.}\} \rangle$  in Lemma 21.0.25.

By Theorem 16.1.1  $(N_\ell, \vartheta) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$  for all  $n < \ell < 2n$  and  $\vartheta$  in general position. It follows that at least one of the periods  $(U_1^a, \psi_1^a)$  is not in  $\mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$ . It follows from theorem 16.2.1 that  $(U_1^a, \psi_1^a)$  vanishes for  $a$  incompatible with  $\chi$ , so it must not vanish for some  $a$  compatible with  $\chi$ . This establishes genericity (and hence nontriviality) of the corresponding descent module  $DC_\omega^a(\tau)$ . The spaces  $DC_\omega^a$  for  $a$  compatible with  $\chi$  may all be identified with one another via suitable isomorphisms among the groups  $G_{2n}^a$ , and so  $DC_\omega^a$  is nonzero and generic for all  $a$  compatible with  $\chi$ .

## (2) **cuspidality:**

Turning to cuspidality, we prove in the appendices an identity relating:

- Constant terms on  $G_{2n}^a$ ,
- Descent periods in  $G_{4n+1}$ ,
- Constant terms on  $G_{4n+1}$ ,
- Descent periods on  $G_{4n-2k+1}$ , embedded in  $G_{4n}$  as a subgroup of a Levi.

To formulate the exact relationship we introduce some notation for the maximal parabolics of  $\mathrm{GSpin}$  groups.

The group  $G_{4n+1}$  has one standard maximal parabolic having Levi  $GL_i \times G_{4n-2i+1}$  for each value of  $i$  from 1 to  $2n$ . Let us denote the unipotent radical of this parabolic by  $V_i$ . We denote the trivial character of any unipotent group by  $\mathbf{1}$ .

For any square class  $\mathbf{a}$ , the group  $G_{2n}^{\mathbf{a}}$  has one standard maximal parabolic having Levi  $GL_k \times G_{2n-2k}^{\mathbf{a}}$  for each value of  $k$  from 1 to  $n-2$ . We denote the unipotent radical of this parabolic by  $V_k^{2n}$ . The split group  $G_{2n} = G_{2n}^\square$  also has two parabolics with Levi isomorphic to  $GL_n \times GL_1$ . One has the property that  $e_{n-1} - e_n$  is a root of the Levi, and the other does not. Let us denote the unipotent radical of this first parabolic by  $V_n^{2n}$ . Then the unipotent radical of the other is  ${}^\dagger V_n^{2n}$ , where  ${}^\dagger$  is the outer automorphism of  $G_{2n}$  which reverses the last two simple roots while fixing the others. In a nonsplit quasisplit form of  $G_{2n}$ , there is a parabolic subgroup with Levi isomorphic to the product of  $GL_{n-1}$  and a nonsplit torus which is maximal. (The corresponding parabolic in the split case is not maximal.) We denote its unipotent radical by  $V_{n-1}^{2n}$ .

We prove in Lemma 21.0.26 that, for  $1 \leq k \leq n-1$ ,  $(V_k^{2n}, \mathbf{1}) \circ (N_n, \Psi_n^a)$  is contained in

$$\langle (N_{n+k}, \Psi_{n+k}), \{(N_{n+j}, \Psi_{n+j}^a)^{(4n-2k+2j+1)} \circ (V_{k-j}, \mathbf{1}) : 1 \leq j < k\} \rangle,$$

where  $(N_{n+j}, \Psi_{n+j}^a)^{(4n-2k+2j+1)}$  denotes the descent period, defined as above, but on the group  $G_{4n-2k+2j+1}$ , embedded into  $G_{4n+1}$  as a component of the Levi with unipotent radical  $V_{k-j}$ .

Now suppose that  $a$  is a square. Then both  $(V_n^{2n}, \mathbf{1}) \circ (N_n, \Psi_n^a)$  and  $({}^\dagger V_n^{2n}, \mathbf{1}) \circ (N_n, \Psi_n^a)$  are in

$$\langle (N_{2n}, \Psi_{2n}), \{(N_{n+j}, \Psi_{n+j}^a)^{(2n+2j+1)} \circ (V_{n-j}, \mathbf{1}) : 1 \leq j < n\} \rangle.$$

Indeed, the two periods are actually conjugate in  $G_{4n+1}$ , so it suffices to consider only one of them.

By Theorem 16.1.1  $(N_{n+k}, \Psi_{n+k}^a) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$  for  $k = 1$  to  $n$ . Furthermore, for  $k, j$  such that  $1 \leq j < k \leq n$  the function  $E(f)(s)^{(V_{k-j}, \mathbf{1})}$  may be expressed in terms of Eisenstein series on  $GL_{k-j}$  and  $G_{4n-2k+2j}$ , using Proposition II.1.7 (ii) of [MW1]. What we require is the following:

**Lemma 16.4.1.** *For all  $f \in V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau \boxtimes \omega)$*

$$E_{-1}(f)^{(V_{k-j}, \mathbf{1})} \Big|_{G_{4n-2k+2j+1}(\mathbb{A})} \in \bigoplus_S \mathcal{E}_{-1}(\tau_S, \omega),$$

where the sum is over subsets  $S$  of  $\{1, \dots, r\}$  such that  $\sum_{i \in S} 2n_i = 2n - k + j$ , and, for each such  $S$ ,  $\mathcal{E}_{-1}(\tau_S, \omega)$  is the space of functions on  $G_{4n-2k+2j+1}(\mathbb{A})$  obtained by applying the construction of  $\mathcal{E}_{-1}(\tau, \omega)$  to  $\{\tau_i : i \in S\}$ , instead of  $\{\tau_i : 1 \leq i \leq r\}$ .

Once again, this is immediate from [MW1] Proposition II.1.7 (ii).

Applying Theorem 16.1.1, with  $\tau$  replaced by  $\tau_S$  and  $2n$  by  $2n - k + j$ , we deduce

$$(N_{n+j}, \Psi_{n+j})^{(4n-2k+2j+1)} \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau_S, \omega)) \quad \forall S,$$

and hence  $(N_{n+j-1}, \Psi_{n+j-1})^{(4n-2k+2j)} \circ (V_{k-j}, \mathbf{1}) \in \mathcal{U}^\perp(\mathcal{E}_{-1}(\tau, \omega))$ . This shows that any nonzero function appearing in any of the spaces  $DC_\omega^a(\tau)$  must be cuspidal. Such a function is also easily seen to be of uniformly moderate growth, being the integral of an automorphic form over a compact domain. In addition, such a function is easily seen to have central character  $\omega$ , and any function with these properties is necessarily square integrable modulo the center ([MW1] I.2.12). It follows that each of the spaces  $DC_\omega^a(\tau)$  decomposes discretely.

### (3) Verification of weak lifting: unramified parameters:

Now, suppose  $\sigma \cong \otimes_v' \sigma_v$  is an irreducible representation which is contained in  $DC_\omega^a(\tau)$ . Let  $p_\sigma$  denote the natural projection  $DC_\omega^a(\tau) \rightarrow \sigma$ . Once again, by Theorem 15.0.12 (3), the representation  $\mathcal{E}_{-1}(\tau, \omega)$  decomposes discretely. Let  $\pi$  be an irreducible component of  $\mathcal{E}_{-1}(\tau, \omega)$  such that the restriction of  $p_\sigma \circ FC^{\Psi_n^a}$  to  $\pi$  is nontrivial. As discussed previously in the proof of Theorem 16.1.1, at all but finitely many  $v$ ,  $\tau$  is unramified at  $v$  and furthermore,  $\pi_v$  is the unramified constituent  ${}^{un}Ind_{P(F_v)}^{G_{4n+1}(F_v)} \tau_v \boxtimes \omega_v \otimes |\det|_{\frac{1}{v}}^{\frac{1}{2}}$  of  $Ind_{P(F_v)}^{G_{4n+1}(F_v)} \tau_v \boxtimes \omega_v \otimes |\det|_{\frac{1}{v}}^{\frac{1}{2}}$ . If  $v_0$  is such a place, the map  $p_\sigma \circ FC^{\Psi_n^a} \circ i_{\zeta v_0}$ , with  $i_{\zeta v_0}$  defined as in Theorem 16.1.1, factors through  $\mathcal{J}_{N_n, \Psi_n^a} \left( {}^{un}Ind_{P(F_{v_0})}^{G_{4n+1}(F_{v_0})} \tau_v \otimes |\det|_{\frac{1}{v}}^{\frac{1}{2}} \boxtimes \omega_v \right)$ , and gives rise to a  $G_{2n}^a(F_{v_0})$ -equivariant map from this Jacquet-module onto  $\sigma_{v_0}$ .

To pin things down precisely, assume that  $\tau_v$  is the unramified component of  $Ind_{B(GL_{2n})(F_v)}^{GL_{2n}(F_v)} \mu$ , and let  $\mu_1, \dots, \mu_{2n}$  be defined as in the proof of Lemma 14.0.6. By Lemma 14.0.6, we may assume without loss of generality that  $\mu_{2n+1-i} = \omega \mu_i^{-1}$  for  $i = 1$  to  $n-1$ , and that either  $\mu_n = \omega \mu_{n+1}^{-1}$ , or  $\mu_n^2 = \mu_{n+1}^2 = \mu_n \mu_{n+1} \chi_{un} = \omega$  (with  $\chi_{un}$  defined as in the lemma). Furthermore, suppose that  $\chi_v$  is the local component at  $v$  of the global quadratic character obtained from  $\tau$  and  $\omega$  as in the statement of the theorem. Then either  $\chi_v$  is trivial and  $\mu_n = \omega \mu_{n+1}^{-1}$ , or  $\chi_v = \chi_{un}$  and  $\mu_n^2 = \mu_{n+1}^2 = \mu_n \mu_{n+1} \chi_{un} = \omega$ .

Recall that a basis for the lattice of  $F$ -rational cocharacters of the maximal torus of  $G_{2n}^a$  fixed in Lemma 16.1.10 is given by

$$\{e_{n+i}^* : 1 \leq i < n\} \cup \{e_0^*\} \cup \{e_n^*, \text{ if } a \text{ is a square}\}.$$

Observe that when  $a$  is not a square in  $F$ , it is a square in  $F_v$  for many unramified  $v$ , and that the cocharacter  $e_n^*$  is  $F_v$ -rational at such  $v$ .

In proposition 20.0.21, we show that in the nonsplit case

$$\mathcal{J}_{N_n, \Psi_n} \left( {}^{un} \text{Ind}_{P(F_v)}^{G_{4n+1}(F_v)} \tau_v \boxtimes \omega_v \otimes |\det|_v^{\frac{1}{2}} \right)$$

is isomorphic as a  $G_{2n}^a(F_v)$ -module to a subquotient of a principal series representation  $\pi_v$  of  $G_{2n}^a(F_v)$  such that the corresponding parameter  $t_{\pi, v}$  maps to the parameter  $t_{\tau, v}$  under  $r$ . In the split case (proposition 20.0.20), we obtain instead a direct sum of two principal series representations, but *both* have parameters which map to  $t_{\tau, v}$ . It follows that  $\tau$  is the weak lift of  $\sigma$  associated to the map  $r$ .

### 17. APPENDIX III: PREPARATIONS FOR THE PROOF OF THEOREM 15.0.12

In this section we review some standard arguments by which the presence or absence of a singularity of an Eisenstein series reduces to the presence or absence of a singularity of a relative rank one intertwining operator.

To do so, we recall the set

$$W(M) := \left\{ w \in W_{G_{4n+1}} \mid \begin{array}{l} w \text{ is of minimal length in } w \cdot W_M \\ wMw^{-1} \text{ is a standard Levi of } G_{4n+1} \end{array} \right\}.$$

It will be convenient and harmless to treat the elements of  $W(M)$  as though they were elements of  $G_{4n+1}(F)$ , rather than repeatedly choose representatives and remark the independence of the choice.

**17.1. Intertwining operators.** For each  $w \in W(M)$ ,  $\underline{s} \in \mathbb{C}^r$ , we define  $P^w$  to be the standard parabolic with Levi  $wMw^{-1}$ . For  $\underline{s}$  such that  $s_r$  and  $s_i - s_{i+1}$ ,  $i = 1$  to  $r - 1$  are all sufficiently large, the integral

$$M(w, \underline{s})f(g) := \int_{U_{\max} \cap w \overline{U_{\max}} w^{-1} (F \backslash \mathbb{A})} f(\underline{s})(w^{-1}ug) \, du$$

converges ([MW1], II.1.6), defining an operator  $M(w, \underline{s})$  from  $V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$  to a space of functions which is easily verified to afford a realization of

$$\text{Ind}_{P^w(\mathbb{A})}^{G_{4n+1}(\mathbb{A})} \left( \left( \bigotimes_{i=1}^r \tau_i \otimes |\det|^{s_i} \right) \boxtimes \omega \right) \circ \text{Ad}(w^{-1}).$$

Here,  $((\bigotimes_{i=1}^r \tau_i \otimes |\det|^{s_i}) \boxtimes \omega) \circ \text{Ad}(w^{-1})$ , denotes the representation of  $wMw^{-1}$  obtained by composing the representation  $(\bigotimes_{i=1}^r \tau_i \otimes |\det|^{s_i} \boxtimes \omega)$  of  $M$  with conjugation by  $w^{-1}$ . We denote this latter space of functions by  $V_w^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ . Then  $M(w, \underline{s})f(g)$  has meromorphic continuation to  $\mathbb{C}^r$ . ([MW1], IV.1.8(b).)

It may be helpful also to review the sorts of singularities which Eisenstein series and intertwining operators have—lying along so-called “root hyperplanes.” (cf. [MW1], IV.1.6) We defer the notion of “root hyperplane” until later. For now, we allow arbitrary hyperplanes in  $\mathbb{C}^r$ , defined by equations of the form  $l(\underline{s}) = c$ , with  $l$  a linear functional  $\mathbb{C}^r \rightarrow \mathbb{C}$  and  $c$  a constant. Then for any bounded open set  $U \subset \mathbb{C}^r$ , there exist a finite number of distinct hyperplanes  $H_1, \dots, H_N$ , which “carry” the singularities of the Eisenstein series and intertwining operators in  $U$ , in the following sense. For

each  $i$  fix  $l_i, c_i$  such that  $H_i = \{\underline{s} \in \mathbb{C}^r \mid l_i(\underline{s}) = c_i\}$ . Then for each  $i$  there is a non-negative integer  $\nu(H_i)$  such that

$$(17.1.1) \quad \prod_{i=1}^N (l_i(\underline{s}) - c_i)^{\nu(H_i)} E(f)(g)(\underline{s})$$

continues to a function holomorphic on all of  $U$ . Covering  $\mathbb{C}^r$  with bounded open sets and taking a union, we obtain an infinite, but *locally* finite, set of hyperplanes which carry all the singularities of the Eisenstein series and intertwining operators. The same hyperplane  $H$  will of course occur more than once. It is easily verified that the minimal exponent  $\nu(H)$  appearing in (17.1.1) is the same each time. Thus we may speak of whether an Eisenstein series or intertwining operator does or does not have a pole along  $H$ , and of the order of the pole.

One may define “analytic/meromorphic continuation” for functions taking values in Fréchet spaces of locally  $L^2$  functions and the like ([MW1] I.4.9, IV.1.3) of functions and operators. In this case, outside of the domain of convergence, one’s functions are defined only up to  $L^2$  equivalence. However, in view of [MW1], I.4.10, one has a unique smooth representative for the class. For us it will be more convenient simply to adopt the convention that when we say the Eisenstein series has a pole along  $H$ , we mean for some  $f, g$ .

Now let us state the relationship between poles of Eisenstein series and intertwining operators, which we prove in section 19.1.

**Proposition 17.1.2.** *For  $f \in V^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ , there exists  $g \in G_{4n+1}(\mathbb{A})$  such that  $E(f)(g)$  has a pole along  $H$  if and only if there exist  $w \in W(M), g' \in G_{4n+1}(\mathbb{A})$  such that  $M(w, \underline{s})f(g')$  has a pole along  $H$ .*

The same construction can be performed with the Levi  $M$  replaced by  $wMw^{-1}$ , yielding an operator

$$M_w(w', w \cdot \underline{s}) : V_w^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega) \rightarrow V_{w'w}^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega),$$

for each  $w' \in W(wMw^{-1})$ . Furthermore, one has for all  $f, g$ , the equality of meromorphic functions

$$M_w(w', w \cdot \underline{s}) \circ M(w, \underline{s})f(g) = M(w'w, \underline{s})f(g)$$

([MW1], II.1.6, IV.4.1). (For now, the reader may think of “ $w \cdot \underline{s}$ ” simply as a notational contrivance. We shall give it a precise meaning below.)

**17.2. Reduction to relative rank one situation.** Next we wish to describe the decomposition of  $w \in W(M)$  as a product of elementary symmetries, as in [MW1] I.1.8. The lattice  $X(Z_M)$  of rational characters of the center of  $M$  has a unique basis  $\{e_0, \varepsilon_1, \dots, \varepsilon_r\}$ , with the property that for each  $i = 1, \dots, m$ , there exists  $j \in \{1, \dots, r\}$  such that the restriction of  $e_i$  as in 4.1 to  $Z_M$  is  $\varepsilon_j$ . The set of restrictions of positive roots of  $G_{4n+1}$  to  $Z_M$  is

$$(17.2.1) \quad \{0\} \cup \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq r\} \cup \{\varepsilon_i : 1 \leq i \leq r\} \cup \{\varepsilon_i + \varepsilon_j : 1 \leq i < j \leq r\} \cup \{2\varepsilon_i : 1 \leq i \leq r\}.$$

Let  $\Phi^+(Z_M)$  denote the set obtained by excluding zero. For  $\alpha \in \Phi^+(Z_M)$ , and  $w \in W(M)$ , one may say “ $w\alpha > 0$ ” or “ $w\alpha < 0$ ” without ambiguity. We say an element of  $\Phi^+(Z_M)$  is indivisible if it is not of the form  $2\varepsilon_i$ .

Each element  $w \in W(M)$  can be decomposed as a product  $s_{\alpha_1} \dots s_{\alpha_\ell}$  of elementary symmetries as in [MW1] I.1.8. The element  $s_{\alpha_\ell}$  will be in  $W(M)$ , while  $s_{\alpha_{\ell-1}}$  will be in  $W(s_{\alpha_\ell} M s_{\alpha_\ell}^{-1})$  and so on. Each is labeled with the unique indivisible restricted root (for the operative Levi) which it reverses. That is  $\{\alpha \in \Phi^+(Z_M) : s_{\alpha_\ell} \alpha < 0\} = \{\alpha_\ell\}$ , or  $\{\alpha_\ell, 2\alpha_\ell\}$  and in the latter case  $\alpha_\ell = \varepsilon_r$ . (Cf. [MW1] I.1.8.)

Let  $w = s_{\alpha_1} \dots s_{\alpha_\ell}$  be a minimal-length decomposition into elementary symmetries, and put  $w_i = s_{\alpha_{i+1}} \dots s_{\alpha_\ell}$ . Then

$$\{\alpha \in \Phi^+(Z_M), \text{ indivisible} \mid w\alpha < 0\} = \{w_i^{-1}\alpha_i \mid 1 \leq i \leq \ell\}$$

and  $\ell$  is the cardinality of this set (i.e., there is no repetition). Combining this discussion with that of the previous paragraphs, we obtain a decomposition of  $M(w, \underline{s})$  as a composite of intertwining operators  $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})$ , each corresponding naturally to one of the elements of  $\{\alpha \in \Phi^+(Z_M), \text{ indivisible} \mid w\alpha < 0\}$ .

Let  $\det_i$  denote the rational character  $(g, \alpha) \mapsto \det g_i$  of  $M$ . Then  $\{e_0, \det_1, \dots, \det_r\}$  is a basis for the lattice  $X(M)$  of rational characters of  $M$ . Here, the character  $e_0$  of  $T$  introduced in section 4.1 has been identified with a character of  $M$  as in 13.1.1. Let  $\{e_0^*, \det_1^*, \dots, \det_r^*\}$  be the dual basis of the dual lattice. Again,  $e_0^*$  is the same as in section 4.1. Elements of  $X(M)$  may be paired with elements of  $X^\vee(T)$  defining a projection from  $X^\vee(T)$  onto the dual lattice. For each  $i = 1, \dots, m$ , there exists unique  $j \in \{1, \dots, r\}$  such that  $e_i^*$  maps to  $\det_j^*$ . If  $\alpha$  is a root, then the projection of the coroot  $\alpha^\vee$  to the dual lattice of  $X(M)$  depends only on the restriction of  $\alpha$  to  $Z_M$ , and the correspondence is as follows:

$$\begin{aligned} 0 &\leftrightarrow 0, \\ \varepsilon_i - \varepsilon_j &\leftrightarrow \det_i^* - \det_j^*, \\ \varepsilon_i + \varepsilon_j &\leftrightarrow \det_i^* + \det_j^* - e_0^*, \\ \varepsilon_i &\leftrightarrow 2\det_i^* - e_0^*, \\ 2\varepsilon_i &\leftrightarrow 2\det_i^* - e_0^*. \end{aligned}$$

We denote the element corresponding to  $\alpha \in \Phi^+(Z_M)$  by  $\alpha^\vee$  (in agreement with [MW1], I.1.11).

We may identify  $\underline{s} \in \mathbb{C}^r$  with

$$\sum_{i=1}^r \det_i \otimes s_i \in X(M) \otimes_{\mathbb{Z}} \mathbb{C}.$$

This is compatible with [MW1], I.1.4. Restriction of functions gives a natural injective map  $X(M) \rightarrow X(T)$ , and hence  $X(M) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow X(T) \otimes_{\mathbb{Z}} \mathbb{C}$ , which we use to identify the first space with a subspace of the second. This gives the notation  $w \cdot \underline{s}$  a precise meaning, as an element of  $X(wMw^{-1}) \otimes_{\mathbb{Z}} \mathbb{C}$ , which is compatible with the usage above. In addition, it gives a “meaning” to the set

$$\{s_i - s_j\} \cup \{s_i + s_j\} \cup \{2s_i\},$$

of linear functionals on  $\mathbb{C}^r$ , identifying each with an element of  $\Phi^+(Z_M)$ . Formally,

**Definition 17.2.2.** *A root hyperplane (relative to the Levi  $M$ ) is a hyperplane of the form*

$$H = \{s \in \mathbb{C}^r \mid \langle \alpha^\vee, \underline{s} \rangle = c\}$$

*for some  $\alpha \in \Phi^+(Z_M)$  which is indivisible, and some  $c \in \mathbb{C}$ . We say that the hyperplane  $H$  is associated to the root  $\alpha$ , which is uniquely determined.*

The next main statement is

**Proposition 17.2.3.** *Let  $w = s_{\alpha_1} \dots s_{\alpha_\ell}$  be any decomposition of minimal length, and for each  $i$  let  $w_i = s_{\alpha_{i+1}} \dots s_{\alpha_\ell}$ . Then the set of poles of  $M(w, \underline{s})$  is the disjoint union of the sets of poles of the operators  $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})$ . A pole of  $M(w, \underline{s})$  comes from  $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})$  if and only if it is associated to  $w_i^{-1}\alpha_i$ . Furthermore, if  $\{\underline{s} \in \mathbb{C}^r \mid \langle \alpha^\vee, \underline{s} \rangle = c\}$  is a pole of  $M(w, \underline{s})$ , then  $c \neq 0$ .*

This is proved in section 19.2.

18. APPENDIX IV: PROOF OF THEOREM 15.0.12

- (1) We now prove Item (1). A root hyperplane passing through  $\frac{1}{2}$  is defined by an equation of one of three forms:  $s_i = \frac{1}{2}$ ,  $s_i + s_j = 1$ , or  $s_i - s_j = 0$ . The third kind can not support singularities of the Eisenstein series. The first two can, but by [MW1]IV.1.11 (c), they will be without multiplicity, and so the factor of

$$\prod_{i \neq j} (s_i + s_j - 1) \prod_{i=1}^r (s_i - \frac{1}{2})$$

will take care of them.

The operators corresponding to elementary symmetries are called relative rank one because they could be defined without reference  $G_{4n+1}$ , considering  $M$  instead as a maximal Levi of another Levi subgroup  $M_\alpha$  of  $G_{4n+1}$ , having semisimple rank one greater than that of  $M$ . Furthermore, in a suitable sense, the relative rank one operator only “lives on one component of  $M_\alpha$ ,” which will allow us to deduce the general case of (2) from the case  $r = 1$  and a similar fact about intertwining operators on  $GL_n$ . Let us make this more precise.

Fix  $\alpha \in \Phi^+(Z_M)$ . There is a minimal Levi subgroup  $M_\alpha$  of  $G_{4n+1}$  containing  $M$  such that  $\alpha$  is the restriction of a root of  $M_\alpha$ . (It is standard iff  $\alpha$  is the restriction of a simple root.) Fix  $w \in W(M)$  such that  $w\alpha < 0$ , and a decomposition  $w = s_{\alpha_1} \dots s_{\alpha_\ell}$  of  $w$  as into elementary symmetries, which is of minimal length. For some unique  $i$ , we have  $\alpha = w_i^{-1}\alpha_i$ , where  $w_i$  is as above. Then  $w_i M_\alpha w_i^{-1}$  is a standard Levi of  $G_{4n+1}$ . Different choices of decomposition give different (even conjugate) embeddings of the same reductive group into  $G_{4n+1}$  as a standard Levi.

If  $\alpha = \varepsilon_j - \varepsilon_k$ , or  $\varepsilon_j + \varepsilon_k$ , then  $M_{\alpha_i}$  is isomorphic to  $GL_{2(n_j+n_k)} \times \prod_{l \neq j,k} GL_{2n_l} \times GL_1$ . while if  $\alpha = \varepsilon_j$ , it is isomorphic to  $G_{4n_j+1} \times \prod_{k \neq j} GL_{2n_k}$ . Let  $G'$  denote  $GL_{2(n_j+n_k)}$  or  $G_{4n_j+1}$  as appropriate and let  $\iota$  be a choice of isomorphism with the “new” factor. Then  $\iota^{-1}(\iota(G') \cap P^{w_i})$  is a maximal parabolic subgroup  $P' = M'U'$  of  $G'$ , and  $\sigma := (\bigotimes_{i=1}^r \tau \otimes \omega) \circ Ad(w_i) \circ \iota$ , is an irreducible unitary cuspidal automorphic representation of  $M'(\mathbb{A})$ . The map  $\iota$  also induces a linear projection

$$\iota_* : X(w_i M w_i^{-1}) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow X(M') \otimes_{\mathbb{Z}} \mathbb{C}.$$

(Recall that we have agreed to think of  $w_i \cdot \underline{s}$  as an element of the former space.)

Following, [MW1] I.1.4, define  $m^\mu$  for  $m \in M'(\mathbb{A})$  and  $\mu$  in  $X(M') \otimes_{\mathbb{Z}} \mathbb{C}$ , by stipulating that  $m^\mu = |\chi(m)|^s$  if  $\mu = \chi \otimes s$  and  $m^{\mu_1 + \mu_2} = m^{\mu_1} m^{\mu_2}$ .

The set  $W_{G'}(M')$ , defined analogously to  $W(M)$  above, contains a unique nontrivial element. It is the elementary symmetry  $s_\beta$  associated to the restriction to  $Z_{M'}$  of the unique simple root of  $G'$  which is not a root of  $M'$ . The map  $\iota$  identifies  $s_\beta$  with  $s_{\alpha_i}$ .

For  $\mu \in X(M') \otimes_{\mathbb{Z}} \mathbb{C}$ , let  $V^{(1)}(\mu, \sigma)$  denote

$$\{h : G'(\mathbb{A}) \rightarrow V_\sigma, \text{ smooth} \mid h(mg')(m') = h'(g')(m'm)m^{\mu+\rho_{P'}} \quad m, m' \in M'(\mathbb{A}), g' \in G'(\mathbb{A})\},$$

$$V^{(2)}(\mu, \sigma) = \{h : G'(\mathbb{A}) \rightarrow \mathbb{C}, \text{ smooth} \mid h(g')(e) \in V^{(1)}(\mu, \sigma)\}.$$

There is a standard intertwining operator  $M(s_\beta, \mu) : V^{(2)}(\mu, \sigma) \rightarrow V_{s_\beta}^{(2)}(\mu, \sigma)$ . One has the identity

$$M_{w_{i-1}}(s_{\alpha_i}, w_i \cdot \underline{s}) f(\iota(h)g) = M(s_\beta, \mu) f(\iota(h)g).$$

That is, if  $p_g$  denotes the map

$$V_{w_i}^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega) \rightarrow V^{(2)}(\mu, \sigma)$$



corresponding to evaluation at  $\iota(h)g$  for a fixed  $g$ , then, for all  $g$ , the following diagram commutes:

$$\begin{array}{ccc} V_{w_i}^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega) & \xrightarrow{M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})} & V_{w_{i-1}}^{(2)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega) \\ p_g \downarrow & & p_g \downarrow \\ V^{(2)}(\iota_*(w_i \cdot \underline{s} + \rho_{P_{\alpha_i}}), \sigma) & \xrightarrow{M(s_{\beta}, \iota_*(w_i \cdot \underline{s} + \rho_{P_{\alpha_i}}))} & V_{s_{\beta}}^{(2)}(\iota_*(w_i \cdot \underline{s} + \rho_{P_{\alpha_i}}), \sigma). \end{array}$$

Hence  $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})$  has a pole along a root hyperplane associated to  $\alpha$  iff  $M(\iota_*(w_i \cdot s + \rho_{P_{\alpha_i}}), \sigma)$  does.

Since the set of poles of  $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})$  is equal to the set of poles of  $M(w, \underline{s})$  along hyperplanes associated to  $\alpha$ , it is independent of the choice of decomposition  $w = s_{\alpha_1} \dots s_{\alpha_\ell}$ . Hence, for each  $\alpha \in \Phi^+(Z_M)$ , we may use a decomposition tailored to that  $\alpha$ .

First suppose  $\alpha = \varepsilon_j - \varepsilon_k$ . One may choose a decomposition so that  $w_i$  corresponds to the permutation matrix in  $GL_{2n}$  (identified with a subgroup of the Siegel Levi) which moves the  $j$ th block of  $M$  up so that it is immediately after the  $i$ th, and otherwise preserves order. It is then easily verified that  $\sigma = \tau_i \otimes \tau_j$  and

$$\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}^{\iota_*(w_i \cdot s) + \rho_{P_{\alpha_i}}} = |\det h_1|^{s_i + \kappa} |\det h_2|^{s_j + \kappa},$$

where  $\kappa = \sum_{k>i, k \neq j} n_k - \sum_{k< i} n_k + n$ .

Next suppose  $\alpha = 2\varepsilon_j$ . Then we choose a decomposition so that  $w_i$  is in the Weyl group of  $GL_{2n}$ , and moves the  $j$ th block to be last, otherwise preserving order. Then one easily verifies that  $\sigma$  is the representation  $\tau_j \boxtimes \omega$  of the Siegel Levi of  $G_{4n_j}$ , and that, for  $(g', \alpha)$  in the Siegel Levi of  $G_{4n_j}$ ,

$$(g', \alpha)^{\iota_*(w_i \cdot \underline{s} + \rho_{P_{\alpha_i}})} = |\det g'|^{s_j}.$$

Finally, suppose  $\alpha = \varepsilon_j + \varepsilon_k$ . Then we choose a decomposition so that  $w_i$  that projects to a permutation matrix in  $SO_{4n+1}$  of the form

$$\begin{pmatrix} I & & & \\ & I & & \\ & & I & \\ I & & & 1 \end{pmatrix},$$

with the off-diagonal blocks being  $2n_j \times 2n_j$ , and the first block being  $\sum_{k=1}^i 2n_k$ . We deduce from Corollary 13.2.5 that  $\sigma = \tau_i \otimes (\tilde{\tau}_j \otimes \omega)$ , and from Lemma 13.2.4 that

$$\begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}^{\iota_*(w_i \cdot s) + \rho_{P_{\alpha_i}}} = |\det h_1|^{s_i + \kappa} |\det h_2|^{-s_j + \kappa},$$

where  $\kappa$  is as before.

(2) Item (2) follows from

**Proposition 18.0.4.** *Let  $w$  denote the unique nontrivial element of  $W(M)$ , in the case when  $M$  is the Levi of the Siegel parabolic of  $G_{2m+1}$ . Let  $\tau$  be a cuspidal representation of  $GL_m$ . Then  $M(w, s)f(g)$  has a pole at  $s = \frac{1}{2}$  for some  $f \in \text{Ind}_{P(\mathbb{A})}^{G_{2m+1}(\mathbb{A})}(\tau \otimes |\det|^s) \boxtimes \omega$ , and  $g \in G_{2m+1}(\mathbb{A})$  if and only if  $\tau$  is  $\omega^{-1}$ -orthogonal*

**Remark 18.0.5.** Of course we are only interested in the case  $m = 2n$ . Furthermore, since we assume  $\omega$  is not the square of another Hecke character, it follows that  $\tau$  can be  $\omega^{-1}$ -orthogonal only if  $m$  is even. However, the proof of this proposition is “blind to” the parity of  $m$ .

**Proposition 18.0.6.** Let  $P = MU$  be a maximal standard parabolic of  $GL_n$  such that  $M \cong GL_k \times GL_{n-k}$ . Let  $f$  be an element of  $\text{Ind}_{P(\mathbb{A})}^{GL_n(\mathbb{A})}(\tau_1 \otimes |\det|^{s_1}) \otimes (\tau_2 \otimes |\det|^{s_2})$ . Let  $w$  be the unique nontrivial element of  $W(M)$ . Then  $M(w, \underline{s})f(g)$  is singular along the hyperplane  $s_1 - s_2 = 1$  for some  $f, g$  iff  $n = 2k$  and  $\tau_2 \cong \tau_1$ .

We defer the proofs to the section 19.

Now, we assume (15.0.14) holds and prove the remaining part of the theorem. Let  $N(\underline{s}) = \prod_{i=1}^r (s_i - \frac{1}{2})$ .

- (3) Item (3) follows from [MW1] I.4.11. The constant term of  $E(f)$  along a parabolic  $P' = M'U'$  has nontrivial cuspidal component iff  $M'$  is conjugate to  $M$ . ([MW1] IV.1.9 (b)(ii)). For such  $P'$  it is equal to

$$\sum_{w \in W(M), wMw^{-1} = M'} M(w, \underline{s})f(g).$$

Take  $w \in W(M)$ , such that  $wMw^{-1} = M'$ . If  $w \cdot \varepsilon_i > 0$  for some  $i$ , then  $M(w, \underline{s})f(g)$  does not have a pole at  $s_i - \frac{1}{2}$ , and hence  $N(\underline{s})M(w, \underline{s})f(g)$  vanishes at  $\frac{1}{2}$ . On the other hand, if  $w \cdot \varepsilon_i < 0$  for all  $i$ , then  $M(w, \underline{s})f(g)$  satisfies the criterion of [MW1] I.4.11.

It follows from [MW1] IV.1.9 (b)(i) applied to  $N(\underline{s})E(f)$  (which is valid by [MW1] IV.1.9 (d)) that the residue is an automorphic form.

- (4) To complete the proof of Item (4), let  $\rho(g)$  denote right translation. It is clear that for values of  $s$  in the domain of convergence,  $N(\underline{s})E(\rho(g)f)(\underline{s}) = N(\underline{s})\rho(g)(E(f)(\underline{s}))$ . By uniqueness of analytic continuation, the equality also holds at values of  $s$  where both sides are defined by analytic continuation, including  $\frac{1}{2}$ . The action of the Lie algebra at the infinite places is handled similarly.

Next we consider the constant term of  $E(f)$  along the Siegel parabolic. By [MW1] II.1.7(ii) it may be expressed in terms of  $GL_{2n}$  Eisenstein series, formed using the functions  $M(w, \underline{s})f$ , corresponding to those  $w \in W(M)$  such that  $w^{-1}(e_i - e_{i+1}) > 0$  for all  $i$ . (Note: we proved in Lemma 13.2.2 that  $wMw^{-1}$  is contained in the Siegel Levi for every  $w \in W(M)$ .) When we pass to  $E_{-1}(f)$ , the term corresponding to  $w$  only survives if  $w \cdot \varepsilon_i < 0$  for all  $i$ . This condition picks out a unique element,  $w_0$ . It is the shortest element of  $W_{GL_{2n}} \cdot w_\ell \cdot W_{GL_{2n}}$ , where  $w_\ell$  is the longest element of  $W_{G_{4n+1}}$ , and we have identified  $GL_{2n}$  with a subgroup of the Siegel Levi as usual. Via corollary 13.2.5 one finds that

$$\left(\bigotimes_{i=1}^r \tau_i \boxtimes \omega\right) \circ \text{Ad}(w_0) = \left(\bigotimes_{i=1}^r (\tilde{\tau}_{r+1-i} \otimes \omega) \boxtimes \omega\right) = \left(\bigotimes_{i=1}^r \tau_{r+1-i} \boxtimes \omega\right).$$

For  $f \in V^{(2)}(\bigotimes_{i=1}^r \tau_i \boxtimes \omega, \frac{1}{2})$ ,  $M(w_0, \frac{1}{2})f|_{GL_{2n}(\mathbb{A})}$  is an element of the analogue of  $V^{(2)}(\bigotimes_{i=1}^r \tau_i \boxtimes \omega, \underline{s})$ , for the induced representation

$$\text{Ind}_{\bar{P}^0(\mathbb{A})}^{GL_{2n}(\mathbb{A})} \left( \bigotimes_{i=1}^r \tau_{r+1-i} \otimes |\det|^{n-\frac{1}{2}} \right) = |\det|^{n-\frac{1}{2}} \otimes \tau$$

of  $GL_{2n}$ . Here  $\bar{P}^0 = GL_{2n} \cap P^{w_0}$ , and  $\tau = \tau_1 \boxplus \dots \boxplus \tau_r$ . Furthermore, since this representation is irreducible, it may be regarded as an arbitrary element. Also, we may regard this representation as induced from  $\tau_1, \dots, \tau_r$  in the usual order. Let  $\bar{P}$  denote the relevant parabolic of  $GL_{2n}$ .

The representation  $\tau$  sits inside a fiber bundle of induced representations  $\text{Ind}_{\bar{P}(\mathbb{A})}^{GL_{2n}(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i \otimes |\det_i|^{s_i})$ . For a flat,  $K$ -finite section  $f$  let  $E^{GL_{2n}}(f)(g)(\underline{s})$  be the  $GL_{2n}$  Eisenstein series defined by

$$\sum_{\bar{P}(F) \backslash GL_{2n}(F)} f(\underline{s})(\gamma g)$$

when  $s_i - s_{i+1}$  is sufficiently large for each  $i$ , and by meromorphic continuation elsewhere.

Let  $U_{\max}^{GL_{2n}}$  denote the usual maximal unipotent subgroup of  $GL_{2n}$ , consisting of all upper triangular unipotent matrices. Let  $\psi_W(u) = \psi_0(u_{1,2} + \cdots + u_{m-1,m})$  be the usual generic character.

(5) To complete the proof of Item (5), we must prove that

$$(18.0.7) \quad \int_{U_{\max}^{GL_{2n}}(F \backslash \mathbb{A})} E^{GL_{2n}}(f)(ug)(\underline{0}) \psi_W(u) du \neq 0$$

for some  $f \in \text{Ind}_{\bar{P}(\mathbb{A})}^{GL_{2n}(\mathbb{A})} \bigotimes_{i=1}^r \tau_{r+1-i}$ ,  $g \in GL_{2n}(\mathbb{A})$ , i.e., that the space of  $GL_{2n}$  Eisenstein series  $E^{GL_{2n}}(f)$  is globally  $\psi_W$ -generic. Granted this, (5) follows from [MW1]II.1.7(ii) and the discussion just above.

The following proposition follows from work of Shahidi.

**Proposition 18.0.8.** *We have the following:*

(a)

$$\int_{U_{\max}^{GL_{2n}}(F \backslash \mathbb{A})} E^{GL_{2n}}(f)(ug)(\underline{s}) \psi_W(u) du = \prod_{v \in S} W_v(g_v) \cdot \prod_{v \notin S} W_v^\circ(g_v) \cdot \prod_{i < j} L^S(s_i - s_j + 1, \tau_i \times \tilde{\tau}_j)^{-1},$$

where,

- for each  $v$ ,  $W_v$  is a Whittaker function in the  $\psi_{W,v}$ -Whittaker model of  $\text{Ind}_{\bar{P}(F_v)}^{GL_{2n}(F_v)}(\bigotimes_{i=1}^r \tau_{i,v} \otimes |\det_i|_v^{s_i})$ ,
- $S$  is a finite set of places, depending on  $f$ ,
- for  $v \notin S$ ,  $\tau_v$  is unramified
- for  $v \notin S$ ,  $W_v^\circ$  is the normalized spherical vector in the  $\psi_{W,v}$ -Whittaker model of  $\text{Ind}_{\bar{P}(F_v)}^{GL_{2n}(F_v)}(\bigotimes_{i=1}^r \tau_{i,v} \otimes |\det_i|_v^{s_i})$ .

(b) A flat,  $K$ -finite section  $f$  may be chosen so that, for all  $v \in S$ , the function  $W_v$  is not identically zero at  $\underline{s} = \underline{0}$ .

We briefly review the steps of the proof in section 19.3.

It follows from [Ja-Sh3] Propositions 3.3 and 3.6 that the product of partial  $L$  functions appearing in Proposition 18.0.8 does not have a pole at  $\underline{s} = \underline{0}$  provided the representations  $\tau_1, \dots, \tau_r$  are distinct. This completes the proof of (5).

(6) Finally, Item (6) follows from the functional equation of the Eisenstein series ([MW1]IV.1.10(a)), and the fact that  $\tau$  is equal to an irreducible full induced representation (as opposed to a constituent of a reducible one).

## 19. APPENDIX V: AUXILLIARY RESULTS USED TO PROVE THEOREM 15.0.12

In this appendix we complete the proofs of several intermediate statements used in the proof of Theorem 15.0.12. As far as we know, all of these results are well-known to the experts, but do not appear in the literature in the precise form we need.

**19.1. Proof of Proposition 17.1.2.** First, suppose that a set  $D$  of hyperplanes carries all the singularities of all the intertwining operators  $M(w, \underline{s})f$ . Then it follows from [MW1] II.1.7, IV.1.9 (b) that it carries all the singularities of the cuspidal components of all the constant terms of  $E(f)(g)(\underline{s})$ . By I.4.10, it therefore carries the singularities of the Eisenstein series itself.

On the other hand, it is clear that a set which carries the singularities of the Eisenstein series carries those of all of its constant terms. Thus, what we need to prove is:

**Lemma 19.1.1.** *Fix  $M'$  a standard Levi which is conjugate to  $M$  and  $\alpha \in \Phi^+(Z_M)$ . Let  $H$  be the root hyperplane given by  $\langle \alpha^\vee, \underline{s} \rangle = c$ ,  $c \neq 0$ . Consider the family of functions  $M(w, \underline{s})f$  corresponding to  $\{w \in W(M) | wMw^{-1} = M'\}$ . If any one or them has a pole along  $H$ , then the constant term of the Eisenstein series along  $P'$  does as well. In other words, it is not possible for two poles to cancel one another.*

*Proof.* Clearly, it is enough to prove this under the additional hypothesis that  $M' = M$ .

Let  $A_M^+$  denote the group isomorphic to  $(\mathbb{R}_+^\times)^{r+1}$ , embedded diagonally at the infinite places, which is inside the center of  $M$ .

The Lie algebra of  $A_M^+$  is naturally identified with the real dual of  $X(M) \otimes_{\mathbb{Z}} \mathbb{R}$ . Recall that above we identified  $\underline{s}$  with an element of  $X(M) \otimes_{\mathbb{Z}} \mathbb{C}$ . So, there is a natural pairing  $\langle X, \underline{s} \rangle$ ,  $X \in \mathfrak{a}_M^+$ , given as follows. Write  $\det_i$  for the determinant of the  $i$ th block of an element of  $M$ , regarded as a  $2n \times 2n$  matrix via the identification with  $GL_n \times GL_n$  fixed above. Then we have

$$\prod_{i=1}^r |\det_i \exp(\log y \cdot X)|^{s_i} = y^{\langle X, \underline{s} \rangle}.$$

It follows that

$$|M(w, \underline{s})f(\exp(\log y \cdot X)g)| = y^{\operatorname{Re}(\langle w^{-1}X, \underline{s} \rangle)} \cdot \delta_P^{\frac{1}{2}}(w^{-1} \exp(\log y \cdot X)w) \cdot |M(w, \underline{s})f(g)|.$$

Here  $\delta_P$  is the modular quasicharacter of  $P$ .

Let

$$W_{\text{sing}}(M, H) = \{w \in W(M), wMw^{-1} = M, M(w, \underline{s}) \text{ has a pole along } H\}.$$

Suppose that this set is nonzero. Choose  $w_0 \in W_{\text{sing}}(M, H)$  such that the order of the pole of  $M(w_0, \underline{s})$  is of maximal order. Let  $\nu(H)$  denote the order. Choose  $X \in \mathfrak{a}_M^+$  such that the points  $w^{-1} \cdot X$ ,  $w \in W_{\text{sing}}(M, H)$  are all distinct. Consider the family of functions

$$(\langle \alpha^\vee, \underline{s} \rangle - c)^{\nu(H)} M(w, \underline{s})f(\exp(\log y \cdot X)g), \quad w \in W_{\text{sing}}(M, H).$$

They have singularities carried by a locally finite set of root hyperplanes not containing  $H$ . Assume  $g$  has been chosen so that  $(\langle \alpha^\vee, \underline{s} \rangle - c)^{\nu(H)} M(w_0, \underline{s})f(g) \neq 0$ . For  $\underline{s}$  restricted to an open subset of  $H$  not intersecting any of the singular hyperplanes we obtain a family of holomorphic functions, at least one of which is nonzero. If we further exclude the intersection of  $H$  with the hyperplanes

$$\langle w_1^{-1}X - w_2^{-1}X, \underline{s} \rangle = 0, \quad w_1, w_2 \in W_{\text{sing}}(M, H),$$

(which can not coincide with  $H$  because  $c \neq 0$ ), then at every point  $\underline{s}$ , those functions which are nonzero all have distinct orders of magnitude as functions of  $y$ . Hence they can not possibly cancel one another.  $\square$

**19.2. Proof of Lemma 17.2.3.** Regarding  $w_i \cdot \underline{s} + \rho_{P_{\alpha_i}}$  as an element of  $X(w_i M w_i^{-1}) \otimes_{\mathbb{Z}} \mathbb{C}$ , we may decompose it as  $\mu_1 + \langle \alpha_i^\vee, w_i \cdot \underline{s} \rangle \tilde{\alpha}_i$ , where  $\tilde{\alpha}_i$  is defined by the property that

$$\langle \alpha^\vee, \tilde{\alpha}_i \rangle = \delta_{\alpha, \alpha_i}, \quad \text{for } \alpha \in \Phi^+(Z_{w_i M w_i^{-1}}).$$

Then it follows easily from the definitions that  $\mu_1$  is in the image of the natural projection  $X(M_{\alpha_i}) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow X(w_i M w_i^{-1}) \otimes_{\mathbb{Z}} \mathbb{C}$  corresponding to restriction of characters of  $M_{\alpha_i}(\mathbb{A})$  to  $w_i M w_i^{-1}(\mathbb{A})$ .

Take  $f$  a  $K$ -finite flat section of  $\text{Ind}_{P^{G_{4n+1}}(\mathbb{A})}^{G_{4n+1}(\mathbb{A})}(\bigotimes_{j=1}^r \tau_j \otimes |\det_j|^{s_j} \boxtimes \omega) \circ \text{Ad}(w_i^{-1})$ . Then  $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})f$  resides in a finite dimensional subspace of  $\text{Ind}_{P^{G_{4n+1}}(\mathbb{A})}^{G_{4n+1}(\mathbb{A})}(\bigotimes_{j=1}^r \tau_j \otimes |\det_j|^{s_j} \boxtimes \omega) \circ \text{Ad}(w_{i-1}^{-1})$ , corresponding to a finite set of  $K$ -types determined by  $f$ . Write  $M_{w_i}(s_{\alpha_i}, w_i \cdot \underline{s})f$  in terms of a basis of flat  $K$ -finite sections. The coefficients are functions of  $\underline{s}$ , but it follows easily from the integral definition where this is valid, and by meromorphic continuation elsewhere, that in fact they are independent of  $\mu_1$  (which corresponds to a character of  $M_{\alpha_1}(\mathbb{A})$  and may be pulled out of the integration). Thus, they depend only on  $\langle w_i \cdot \underline{s}, \alpha_i^\vee \rangle = \langle \underline{s}, w_i^{-1} \alpha_i^\vee \rangle$ .

The first two assertions are now clear. A proof that  $c \neq 0$  is obtained by a straightforward modification of the opening paragraph of [MW1], IV.3.12.

**19.3. Proof of Proposition 18.0.4.** In this section, we denote by  $V^{(i)}(s, \tau, \omega)$ ,  $i = 1, 2$ , the spaces of functions previously introduced in section 15 as  $V^{(i)}(\underline{s}, \bigotimes_{i=1}^r \tau_i \boxtimes \omega)$ , in the special case when  $r = 1$ .

Let  $\tilde{M}(s)$  denote the analogue of  $M(w, s)$  defined using  $V^{(1)}(s, \tau, \omega)$ . It maps into the space  $V^{(3)}(-s, \tilde{\tau} \otimes \omega, \omega)$  given by

$$\left\{ \tilde{F} : G_{2m+1}(\mathbb{A}) \rightarrow V_\tau, \text{ smooth } \left| \tilde{F}((g, \alpha)h)(g_1) = \omega(\alpha \det g) |\det g|^{-s + \frac{m}{2}} \tilde{F}(h)(g_1 t g^{-1}) \right. \right\}.$$

Fix realizations of the local induced representations  $\tau_v$  and an isomorphism  $\iota : \otimes'_v \tau_v \rightarrow \tau$ . Define, for each  $v$ ,  $V^{(1)}(s, \tau_v, \omega_v)$  to be

$$\left\{ \tilde{F}_v : G_{2m+1}(F_v) \rightarrow V_{\tau_v}, \text{ smooth } \left| \tilde{F}_v((g, \alpha)h) = \omega_v(\alpha) |\det g|_v^{s + \frac{m}{2}} \tau_v(g) \tilde{F}_v(h) \right. \right\},$$

and  $V^{(3)}(s, \tilde{\tau}_v \otimes \omega_v, \omega_v)$  to be

$$\left\{ \tilde{F}_v : G_{2m+1}(F_v) \rightarrow V_{\tau_v}, \text{ smooth } \left| \tilde{F}_v((g, \alpha)h) = \omega_v(\alpha \det g) |\det g|_v^{s + \frac{m}{2}} \tau_v(t g^{-1}) \tilde{F}_v(h) \right. \right\}.$$

Then the formula

$$\tilde{\iota}(\otimes_v \tilde{F}_v)(g) = \iota(\otimes'_v \tilde{F}_v(g_v))$$

defines maps

$$\otimes'_v V^{(1)}(s, \tau_v, \omega_v) \rightarrow V^{(1)}(s, \tau, \omega),$$

$$\otimes'_v V^{(3)}(s, \tilde{\tau}_v \otimes \omega_v, \omega_v) \rightarrow V^{(3)}(s, \tilde{\tau} \otimes \omega_v, \omega),$$

both of which we denote by  $\tilde{\iota}$ .

It is known that each map is, in fact, an isomorphism. For the benefit of the reader we sketch an argument. On pp. 307 of [Sha1] certain explicit elements of (a generalization of)  $V^{(1)}(s, \tau, \omega)$  are constructed as integrals involving matrix coefficients. Using Schur orthogonality, one may check that  $\tilde{F}$  is expressible in this form iff both the  $K$ -module it generates and the  $K \cap M(\mathbb{A})$ -module it generates are irreducible. It is clear that such vectors span the space of all  $K$ -finite vectors. On the other hand the (finite dimensional) space of matrix coefficients of this irreducible representation of  $K$  is spanned by those that factor as a product of matrix coefficients of local representations, and these are clearly in the image of  $\tilde{\iota}$ .

For  $\tilde{F}_v \in V^{(1)}(s, \tau_v, \omega_v)$ , let

$$A_v(s) \tilde{F}_v(g) = \int_{U_w(F_v)} \tilde{F}_v(wug) du.$$

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Then the following diagram commutes

$$\begin{array}{ccc} \otimes'_v V^{(1)}(s, \tau_v, \omega_v) & \xrightarrow{A(s)} & \otimes'_v V^{(1)}(-s, \tau_v, \omega_v) \\ \tilde{\iota} \downarrow & & \downarrow \tilde{\iota} \\ V^{(1)}(s, \tau, \omega) & \xrightarrow{\tilde{M}(s)} & V^{(1)}(-s, \tau, \omega) \end{array}$$

with  $A(s) := \otimes_v A_v(s)$ .

Now,  $M(w, s)f(s)$  has a pole (i.e., there exists  $g \in G_{2m+1}(\mathbb{A})$  such that  $M(w, s)f(s)(g)$  has a pole) if and only if  $\tilde{M}(s)\tilde{F}(s)$  has a pole (i.e., there exist  $g \in G_{2m+1}(\mathbb{A})$  and  $m \in M(\mathbb{A})$  such that  $\tilde{M}(s)\tilde{F}(s)(g)(m)$  has a pole), where  $\tilde{F}$  is the element of  $V^{(1)}(s, \tau, \omega)$  such that  $f(g) = \tilde{F}(g)(id)$ .

We wish to show that there exists  $\tilde{F}$  such that this is the case iff  $\tau$  is  $\omega^{-1}$ -orthogonal. Clearly, we may restrict attention to  $\tilde{F}$  of the form  $\tilde{\iota}(\otimes_v \tilde{F}_v)$ .

Recall that for all but finitely many non-archimedean  $v$ , the space  $V_{\tau_v}$  comes equipped with a choice of  $GL_m(\mathfrak{o}_v)$ -fixed vector  $\xi_v^\circ$  used to define the restricted tensor product.

If  $\tilde{F} = \tilde{\iota}(\otimes_v \tilde{F}_v) \in V^{(1)}(s, \tau, \omega)$ , then there is a finite set  $S$  of places, such that if  $v \notin S$  then  $v$  is non-archimedean,  $\tau_v$  is unramified, and  $\tilde{F}_v(s) = \tilde{F}_{(s, \tau_v, \omega_v)}^\circ$  is the unique element of  $V^{(1)}(s, \tau_v, \omega_v)$  satisfying  $\tilde{F}_{(s, \tau_v, \omega_v)}(k) = \xi_v^\circ$  for all  $k \in G_{2m+1}(\mathfrak{o}_v)$ .

Now

$$A_v(s)\tilde{F}_{(s, \tau_v, \omega_v)}^\circ = \frac{L_v(2s, \tau_v, \text{sym}^2 \times \omega_v^{-1})}{L_v(2s+1, \tau_v, \text{sym}^2 \times \omega_v^{-1})} \tilde{F}_{(-s, \tilde{\tau}_v \otimes \omega_v, \omega_v)}^\circ.$$

(A proof of this appears in [L1], albeit not in this precise language. See especially pp. 25-27.) Thus,

$$A(s)\tilde{\iota}(\otimes_v \tilde{F}_v) = \frac{L^S(2s, \tau, \text{sym}^2 \times \omega^{-1})}{L^S(2s+1, \tau, \text{sym}^2 \times \omega^{-1})} \tilde{\iota} \left( \left( \bigotimes_{v \in S} A_v(s)\tilde{F}_v(s) \right) \otimes \left( \bigotimes_{v \notin S} \tilde{F}_{-s, \tilde{\tau}_v \otimes \omega_v, \omega_v} \right) \right).$$

To complete the proof we must show:

- (i):  $A_v(s)$  is holomorphic and nonvanishing (i.e., not the zero operator) on  $\text{Ind}_{P(\mathbb{A})}^{G_{2m}(\mathbb{A})} \tau \otimes |\det|^s \boxtimes \omega$  at  $s = \frac{1}{2}$ , for all  $\tau$ .
- (ii):  $L_v(s, \tau_v, \text{sym}^2 \times \omega_v^{-1})$  is holomorphic and nonvanishing at  $s = 1$ , for all  $\tau_v$ .
- (iii):  $L^S(s, \tau, \text{sym}^2 \times \omega^{-1})$  is holomorphic and nonvanishing at  $s = 2$ .

Item (iii) is covered by Proposition 7.3 of [Kim-Sh]. Items (i) and (ii) are essentially contained in Proposition 3.6, p. 153 of [Asg-Shal]. Since what we need is *part* of the same information, presented differently, we repeat the part of the arguments we are using.

The nonvanishing part of (i) is a completely general fact (i.e., holds at least for any Levi of any split reductive group). For example, the only element of the arguments made on p. 813 of [GRS3] which is particular to the situation they consider there (the Siegel of  $Sp_{4n}$ ) is the precise ratio of  $L$  functions appearing in the constant term.

Similarly, local  $L$  functions never vanish. At a finite prime the local  $L$  function is  $P(q_v^{-s})^{-1}$  for some polynomial  $P$ , while at an infinite prime it is given in terms of the  $\Gamma$  function and functions of exponential type.

We turn to holomorphicity.

**Lemma 19.3.1.** *Let  $\pi$  be any representation of  $GL_m(F_v)$ , which is irreducible, generic, and unitary. Then there exist*

- integers  $k_1, \dots, k_r$  of such that  $k_1 + \dots + k_r = m$ ,
- real numbers  $\alpha_1, \dots, \alpha_r \in (-\frac{1}{2}, \frac{1}{2})$ ,
- discrete series representations  $\delta_i$  of  $GL_{k_i}(F_v)$  for  $i = 1$  to  $r$



such that

$$\pi \cong \text{Ind}_{P_{(k)}(F_v)}^{GL_m(F_v)} \bigotimes_{i=1}^r (\delta_i \otimes |\det_i|^{\alpha_i}).$$

Here  $P_{(k)}$  denotes the standard parabolic of  $GL_m$  with Levi consisting of block diagonal matrices with the block sizes  $k_1, \dots, k_r$  (in that order), and  $\det_i$  denotes the determinant of the  $i$ th block.

**Remark 19.3.2.** In fact, one may prove a much more precise statement, but the above is what is needed for our purposes.

*Proof.* This follows from [Ja-Sh4] and proposition 3.3 of [MS].  $\square$

Continuing with the proof of Proposition 18.0.4, let  $(k) = (k_1, \dots, k_r)$ ,  $\delta = (\delta_1, \dots, \delta_r)$  and  $\alpha = (\alpha_1, \dots, \alpha_r)$  be such that

$$\tau_v \cong \text{Ind}_{P_{(k)}(F_v)}^{GL_m(F_v)} \bigotimes_{i=1}^r (\delta_i \otimes |\det_i|^{\alpha_i}),$$

and let  $\tilde{P}_{(k)}$  denote the standard parabolic of  $G_{2m}$  which is contained in the Siegel parabolic  $P$  such that  $\tilde{P}_{(k)} \cap M = P_{(k)}$ . Then

$$\text{Ind}_{P(F_v)}^{G_{2m}(F_v)} \tau_v \otimes |\det|_v^s \boxtimes \omega_v^s \cong \text{Ind}_{\tilde{P}_{(k)}(F_v)}^{G_{2m}(F_v)} \bigotimes_{i=1}^r (\delta_i \otimes |\det_i|_v^{s+\alpha_i}) \boxtimes \omega_v.$$

This family (as  $s$  varies) of representations lies inside the larger family,

$$\text{Ind}_{\tilde{P}_{(k)}(F_v)}^{G_{2m}(F_v)} \bigotimes_{i=1}^r (\delta_i \otimes |\det_i|^{s_i}) \boxtimes \omega_v \quad s = (s_1, \dots, s_r) \in \mathbb{C}^r,$$

and our intertwining operator  $A_v(s)$  is the restriction, to the line  $s_i = s + \alpha_i$  of the standard intertwining operator for this induced representation, which we denote  $A_v(\underline{s})$ . This operator is defined, for all  $\text{Re}(s_i)$  sufficiently large, by the same integral as  $A_v(s)$ .

A result of Harish-Chandra says that “ $\text{Re}(s_i)$  sufficiently large” can be sharpened to “ $\text{Re}(s_i) > 0$ .” (This is because all  $\delta_i$  are discrete series, although tempered would be enough.) This result is given in the  $p$ -adic case as [Sil] Theorem 5.3.5.4, and in the Archimedean case, [Kn] Theorem 7.22, p. 196.

Hence, the integral defining  $A_v(s)$  converges for  $s > \max_i(-\alpha_i)$ , and in particular converges at  $\frac{1}{2}$ .

From the relationship between the local  $L$  functions and the so-called local coefficients, it follows that the local  $L$  functions are also holomorphic in the same region. For this relationship see [Sha3] for the Archimedean case and [Sha2], p. 289 and p. 308 for the non-Archimedean case.

This completes the proof of (i) and (ii).

**19.4. Proof of Proposition 18.0.6.** The proof is the same as the previous proposition, except that the ratio of partial  $L$  function which emerges from the intertwining operators at the unramified places is

$$\frac{L^S(s_1 - s_2, \tau_1 \times \tilde{\tau}_2)}{L^S(s_1 - s_2 + 1, \tau_1 \times \tilde{\tau}_2)}.$$

Convergence of local  $L$  functions and intertwining operators at  $s_1 - s_2 = 1$  follows again from Lemma 19.3.1. The only difference is the reference for (iii), which in this case is Theorem 5.3 on p. 555 of [Ja-Sh2].

**19.5. Proof of 18.0.8.** As noted, this material is mostly due to Shahidi.

Since the statement is true (with the same proof) for general  $m$ , not only  $m = 2n$ , we prove it in that setting.

In this subsection only, we write  $\tau$  for the irreducible unitary cuspidal representation  $\bigotimes_{i=1}^r \tau_i$  of  $M(\mathbb{A})$  (as opposed to the isobaric representation  $\tau_1 \boxplus \cdots \boxplus \tau_r$ ).

First, observe that the integral in question is clearly absolutely and uniformly convergent, and as such defines a meromorphic function of  $\underline{s}$  for each  $g$  with poles contained in the set of poles of the Eisenstein series itself.

For  $\underline{s}$  in the domain of convergence

$$(19.5.1) \quad \int_{U_{\max}^{GL_m}(F \backslash \mathbb{A})} E^{GL_m}(f)(ug)(\underline{s}) \psi_W(u) du = \int_{U_{w_1}(\mathbb{A}) \cdot U^{w_1}(F \backslash \mathbb{A})} f(\underline{s})(w_1^{-1}ug) \psi_W(u) du,$$

where  $w_1$  is the longest element of  $W_{GL_m}(\bar{M})$  (defined analogously to  $W(M)$  above),  $U_{w_1} = U_{\max}^{GL_m} \cap w_1 \overline{U_{\max}^{GL_m}} w_1^{-1}$  and  $U^{w_1} = U_{\max}^{GL_m} \cap w_1 U_{\max}^{GL_m} w_1^{-1}$ .

Indeed,

$$\bar{P}(F) \backslash GL_m(F) = \coprod_w w^{-1} U_w(F),$$

where the union is over  $w$  of minimal length in  $w W_{\bar{M}}$ . Telescoping, we obtain a sum of terms similar to the right hand side of (19.5.1) for these  $w$ . Let  $U_{\max}^M = M \cap U_{\max}$ . Observe that  $w U_{\max}^M w^{-1} \subset U_{\max}$  for all such  $w$ . The restriction of  $\psi_W$  to  $w U_{\max}^M w^{-1}$  is a generic character iff  $w M w^{-1}$  is a standard Levi. If it is not, the term corresponding to  $w$  vanishes by cuspidality of  $\tau$ .

On the other hand,  $f(w^{-1}ug)$  vanishes if  $w^{-1}U_{\alpha}w$  is contained in the unipotent radical of  $\bar{P}$  (which we denote  $U_{\bar{P}}$ ) for any simple root  $\alpha$ . Here  $U_{\alpha}$  denotes the one-dimensional unipotent subgroup corresponding to the root  $\alpha$ . The element  $w_1$  is the only element of  $W_{GL_m}(\bar{M})$  such that this does not hold for any  $\alpha$ .

Let  $\lambda$  denote the Whittaker functional on  $V_{\tau}$  given by

$$\varphi \mapsto \int_{U_{\max}^M(F \backslash \mathbb{A})} \varphi(u) \psi_W(w_1 u w_1^{-1}) du.$$

Then (19.5.1) equals

$$(19.5.2) \quad \int_{U_{w_1}(\mathbb{A})} \lambda(\tilde{f}(\underline{s})(ug)) \psi_W(u) du,$$

where  $\tilde{f}: GL_m(\mathbb{A}) \rightarrow V_{\otimes \tau_i}$  is given by  $\tilde{f}(g)(m) = f(mg) \delta_{\bar{P}}^{-\frac{1}{2}}$ . (I.e.,  $\tilde{f}$  is the element of the analogue of  $V^{(1)}(\bigotimes_{i=1}^r \tau_i \boxtimes \omega, \underline{s})$ , corresponding to  $f$ .)

For each place  $v$  there exists a Whittaker functional  $\lambda_v$  on the local representation  $\tau_v$  such that  $\lambda(\otimes_v \xi_v) = \prod_v \lambda_v(\xi_v)$ . (A finite product because  $\lambda_v(\xi_v^{\circ}) = 1$  for almost all  $v$ . Cf. [Sha1], §1.2.) The induced representation  $\text{Ind}_{\bar{P}(\mathbb{A})}^{GL_m(\mathbb{A})}(\bigotimes_{i=1}^r \tau_i | \det_i |^{s_i})$  is isomorphic to a restricted tensor product of local induced representations  $\otimes_v' \text{Ind}_{\bar{P}(F_v)}^{GL_m(F_v)}(\bigotimes_{i=1}^r \tau_{i,v} | \det_i |^{s_i})$ . (Cf. section 19.3.) Consider an element  $\tilde{f}$  which corresponds to a pure tensor  $\otimes_v \tilde{f}_v$  in this factorization. So  $\tilde{f}_v(\underline{s})$  is a smooth function  $GL_m(F_v) \rightarrow V_{\otimes \tau_{i,v}}$  for each  $\underline{s}$ . Then (19.5.2) equals

$$(19.5.3) \quad \prod_v \int_{U_{w_1}(F_v)} \lambda_v(\tilde{f}(\underline{s})(u_v g_v)) \psi_W(u_v) du_v,$$

whenever each of the local integrals is convergent, and the infinite product is convergent (cf [Tate2] Theorem 3.3.1). By Propositions 3.1 and 3.2 of [Sha4], all of the local integrals are always convergent. (See also Lemma 2.3 and the remark at the end of section 2 of [Sha3].)

It is an application of Theorem 5.4 of [C-S] that the term corresponding to an unramified nonarchimedean place  $v$  in (19.5.2) is equal to  $W_v^\circ(g_v) \cdot \prod_{i < j} L_v(s_i - s_j + 1, \tau_{i,v} \otimes \tilde{\tau}_{j,v})^{-1}$ . The convergence of the infinite product is then an elementary exercise, as is the main equation in the statement of our present theorem.

The fact that  $f$  may be chosen so that the local Whittaker functions at the places in  $S$  do not vanish follows again from Propositions 3.1 and 3.2 of [Sha4] (see also the remark at the end of section 2 of [Sha3]).

## 20. APPENDIX VI: LOCAL RESULTS ON JACQUET FUNCTORS

In this appendix,  $F$  is a non-archimedean local field of characteristic zero. We denote the ring of integers and its unique maximal ideal by  $\mathfrak{o}$ , and  $\mathfrak{p}$ , respectively, and let  $q_F := \#\mathfrak{o}/\mathfrak{p}$ . The absolute value on  $F$  is normalized so that its image is  $\{q_F^j : j \in \mathbb{Z}\}$ . Also,  $\omega$  is an unramified character of  $F^\times$ ,  $\tau$  is an irreducible unramified principal series representation of  $GL_{2n}(F)$  such that  $\tau \cong \tilde{\tau} \otimes \omega$ , and  $\psi_0$  is a nontrivial additive character of  $F$ .

For simplicity, we assume that the characteristic of the residue field  $\mathfrak{o}/\mathfrak{p}$  is not equal to two. Hence there are four square classes in  $F$ , of which two contain units. If  $\vartheta$  is a character of  $N_\ell(F)$  for  $1 \leq \ell \leq 2n$ , then we may define the square class  $\text{Inv}(\vartheta)$  as in Definition 16.1.4 and it is an invariant which separates orbits of characters in general position. Where convenient, we may restrict attention to those  $\vartheta$  such that  $\text{Inv}(\vartheta)$  contains units, as this condition is satisfied at almost all places by any global character. We also define abstract  $F$ -groups

$$G_{2n}^{\mathbf{a}} \quad \mathbf{a} \in F^\times / (F^\times)^2,$$

and concrete subgroups

$$G_{2n}^a \subset G_{4n+1} \quad a \in F^\times,$$

such that  $G_{2n}^a \cong G_{2n}^{\mathbf{a}} \forall a \in \mathbf{a}$ , as in Definitions 16.1.5, and 16.1.9. The latter is defined using a character  $\Psi_n^a$  given by the same formula as in Definition 16.1.8.

We require the additional technical hypothesis

$$(20.0.4) \quad (B(G_{4n+1}) \cap G_{2n}^a)(F) G_{2n}^a(\mathfrak{o}) = G_{2n}^a(F),$$

which is known (see [Tits], 3.9, and 3.3.2) to hold at all but finitely many non-Archimedean completions of a number field.

Throughout this section we shall express certain characters of reductive  $F$ -groups as complex linear combinations of rational characters. The identification is such that

$$\left( \sum_{i=1}^r s_i \chi_i \right) (h) := \prod_{i=1}^r |\chi_i(h)|^{s_i}.$$

Clearly, the coefficients  $s_1, \dots, s_r$  appearing in this expression are determined by the character at most up to  $(2\pi i)/\log q_F$ . If  $M$  is a Levi, then restriction gives an injective map  $X(M) \rightarrow X(T)$ . We shall frequently abuse notation and denote an element of  $X(M)$  by the same symbol as its restriction to  $T$ . Finally, we let  $\Omega$  denote a complex number such that  $\omega(x) = |x|^\Omega$ .

Lemma 14.0.6 may be reformulated as stating that  $\tau \cong \text{Ind}_{B(GL_{2n})(F)}^{GL_{2n}(F)} \mu$  for an unramified character  $\mu$ , which is of one of the the following two forms:

$$(20.0.5) \quad \mu_1 e_1 + \dots + \mu_n e_n + (\Omega - \mu_n) e_{n+1} + \dots + (\Omega - \mu_1) e_{2n}$$

$$(20.0.6) \quad \mu_1 e_1 + \dots + \mu_{n-1} e_{n-1} + \frac{\Omega}{2} e_n + \left( \frac{\Omega}{2} + \frac{\pi i}{\log q_F} \right) e_{n+1} + (\Omega - \mu_{n-1}) e_{n+2} + \dots + (\Omega - \mu_1) e_{2n}.$$

In either case, by induction in stages,

$${}^{un} \text{Ind}_{P(F)}^{G_{4n+1}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega \cong {}^{un} \text{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)} \mu + \frac{1}{2}(e_1 + \cdots + e_{2n}) + \Omega e_0.$$

(Here  ${}^{un}$  indicates the unramified constituent, and  $P$  the Siegel parabolic of  $G_{4n+1}$ .)

**Remark 20.0.7.** *Because every unramified character is the square of another unramified character, it is possible to express  $\tau$  as a twist of a self-dual representation, and deduce essentially all the results of this section from the “classical,” self-dual case.*

**Lemma 20.0.8.** *If  $\mu$  is of the form (20.0.5), then*

$${}^{un} \text{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)} \mu + \frac{1}{2}(e_1 + \cdots + e_{2n}) + \Omega e_0 \cong {}^{un} \text{Ind}_{P_1(F)}^{G_{4n+1}(F)} \mu'$$

where  $P_1$  is the standard parabolic with Levi isomorphic to  $GL_2^n \times GL_1$ , such that the roots of the Levi are  $e_{2i-1} - e_{2i}$ ,  $i = 1$  to  $n$ , and  $\mu'$  is the rational character of this Levi given by

$$\mu' := \mu_1 \det_1 + \cdots + \mu_n \det_n + \Omega e_0.$$

Here  $\det_i$  denotes the determinant of the  $GL_2$ -factor with unique root  $e_{2i-1} - e_{2i}$ .

*Proof.* Let

$$\tilde{\mu} = \mu + \frac{1}{2}(e_1 + \cdots + e_{2n}) + \Omega e_0 = \sum_{i=1}^n \left( \mu_i + \frac{1}{2} \right) e_i + \sum_{i=1}^n \left( \Omega - \mu_i + \frac{1}{2} \right) e_{n+i} + \Omega e_0.$$

Using the description of the Weyl action in Lemma 13.2.2 it is easily verified that this is in the same orbit as

$$\tilde{\mu}' := \sum_{i=1}^n \left[ \left( \mu_i + \frac{1}{2} \right) e_{2i-1} + \left( \mu_i - \frac{1}{2} \right) e_{2i} \right] + \Omega e_0.$$

By the definition of the unramified constituent, then,

$${}^{un} \text{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)} \tilde{\mu} = {}^{un} \text{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)} \tilde{\mu}'.$$

The lemma now follows from the well known (and easily verified) fact that

$$(20.0.9) \quad {}^{un} \text{Ind}_{B(GL_2)(F)}^{GL_2(F)} \left( \mu + \frac{1}{2} \right) e'_1 + \left( \mu - \frac{1}{2} \right) e'_2 = {}^{un} \text{Ind}_{B(GL_2)(F)}^{GL_2(F)} \left( \mu - \frac{1}{2} \right) e'_1 + \left( \mu + \frac{1}{2} \right) e'_2 = \mu \det,$$

where  $e'_1$  and  $e'_2$  are the usual basis for the lattice of rational characters of the torus of diagonal elements of  $GL_2$ .  $\square$

The next lemma is similar, but slightly more complicated. It makes use of alternative  $\mathbb{Z}$ -bases of the lattices of characters and cocharacters. Specifically,  $\{e_1, \dots, e_{2n-2}, f_1, f_2, f_0\}, \{e_1^*, \dots, e_{2n-2}^*, f_1^*, f_2^*, f_0^*\}$ , where

$$\begin{aligned} e_0 &= -f_1 & e_0^* &= -2f_0^* - f_1^* - f_2^* \\ e_{2n-1} &= -f_0 + f_1 + f_2 & e_{2n-1}^* &= -f_0^* \\ e_{2n} &= f_1 - f_2 & e_{2n}^* &= -f_0^* - f_2^*. \end{aligned}$$

The key feature of these  $\mathbb{Z}$ -bases is as follows. Recall that the group  $G_{4n+1}$  has a unique standard Levi isomorphic to  $GL_2^{n-1} \times G_5$ , with the based root datum of the  $G_5$  component lying in the sublattices spanned by  $\{e_{2n-1}, e_{2n}, e_0\}, \{e_{2n-1}^*, e_{2n}^*, e_0^*\}$ . Now,  $G_5$  and  $GSp_4$  are the same  $F$ -group. When we write the based root datum of this Levi with respect to the new basis, the expression for the  $G_5$  component matches the “standard form” for the based root datum of  $GSp_4$  as in section 4.1. In particular, the character  $f_0$  is the restriction to the torus of  $GSp_4$  of the similitude factor (which is a generator for the rank-one lattice of rational characters of  $GSp_4$ ), and there is a standard Levi, isomorphic to  $GL_2$  such that its unique root is  $f_1 - f_2$ .

**Remarks 20.0.10.** To avoid confusion, let us draw attention the following tricky point: we have defined a notion of “Siegel parabolic” and “Siegel Levi” for  $G_{2n+1}$ , any  $n$ . There is also a well known notion of “Siegel parabolic” and “Siegel Levi” for  $GSp_{2n}$ , any  $n$ , which is very widespread in the literature. The two groups  $G_5$  and  $GSp_4$  happen to coincide, and the two notions of “Siegel parabolic” and “Siegel Levi” do **not**.

**Lemma 20.0.11.** If  $\mu$  is of the form (20.0.6), and  $\tilde{\mu}$  is defined in terms of  $\mu$  as in the proof of Lemma 20.0.8, then

$${}^{un}\mathrm{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)} \tilde{\mu} \cong {}^{un}\mathrm{Ind}_{P_2(F)}^{G_{4n+1}(F)} \mu''$$

where

$$\begin{aligned} \mu'' &= \sum_{i=1}^{n-1} \mu_i \det_i - \frac{\Omega-1}{2} f_0 + \left( -\frac{1}{2} + \frac{\pi i}{\log q_F} \right) \det_0 \\ &= \sum_{i=1}^{n-1} \mu_i \det_i + \frac{\Omega-1}{2} e_{2n+1} - \left( \frac{\Omega}{2} + \frac{\pi i}{\log q_F} \right) \det_0 \end{aligned}$$

where the notation is as follows:  $P_2$  is the standard parabolic with Levi isomorphic to  $GL_2^n \times GL_1$ , such that the roots of the Levi are  $e_{2i-1} - e_{2i}$ ,  $i = 1$  to  $n-1$ , and  $e_{2n}$ . (One might also describe this Levi as  $GL_2^{n-1} \times GL_1 \times GSpin_3$ .) As in Lemma 20.0.8  $\det_i$  denotes the determinant of the  $GL_2$ -factor with unique root  $e_{2i-1} - e_{2i}$ , for  $i = 1$  to  $n-1$ , while  $\det_0$  denotes the determinant of the  $GL_2$  with unique root  $e_{2n} = f_1 - f_2$ .

*Proof.* This time  $\tilde{\mu}$  is in the same Weyl orbit as

$$\begin{aligned} \tilde{\mu}'' &:= \sum_{i=1}^{n-1} \left[ \left( \mu_i + \frac{1}{2} \right) e_{2i-1} + \left( \mu_i - \frac{1}{2} \right) e_{2i} \right] + \left( \frac{\Omega-1}{2} \right) e_{2n-1} + \left( \frac{\Omega-1}{2} + \frac{\pi i}{\log q_F} \right) e_{2n} + \Omega e_0 \\ &= \sum_{i=1}^{n-1} \left[ \left( \mu_i + \frac{1}{2} \right) e_{2i-1} + \left( \mu_i - \frac{1}{2} \right) e_{2i} \right] - \left( \frac{\Omega-1}{2} \right) f_0 + \left( -1 + \frac{\pi i}{\log q_F} \right) f_1 - \frac{\pi i}{\log q_F} f_2. \end{aligned}$$

Using (20.0.9) again, in conjunction with the fact that  $-\frac{\pi i}{\log q_F} f_2$  and  $\frac{\pi i}{\log q_F} f_2$  are the same character, we obtain the lemma.  $\square$

Next, we need a slight extension of this. Let  $P_3$  be the standard parabolic of  $G_{4n+1}$  with Levi isomorphic to  $GL_2^{n-1} \times GSp_4$ . Identify  $GSp_4$  with the component of this Levi, and let  $R = GSp_4 \cap P_2$ . This is the subgroup known in the literature as the “Siegel” parabolic of  $GSp_4$ . When regarded as a parabolic of  $GSpin_5$ , it is the one for which we have introduced the notation  $Q_1 = L_1 N_1$ . Its lattice of rational characters is spanned by  $f_0$  and  $\det_0$ , defined as in Lemma 20.0.11. Let  $\pi_0 = {}^{un}\mathrm{Ind}_{R(F)}^{GSp_4(F)} \left( \frac{1}{2} + \frac{\pi i}{\log q_F} \right) \det_0$ . Extend  $\pi_0$  trivially to a representation of the Levi of  $P_3$ .

**Corollary 20.0.12.**

$${}^{un}\mathrm{Ind}_{B(G_{4n+1})(F)}^{G_{4n+1}(F)} \tilde{\mu}'' \cong {}^{un}\mathrm{Ind}_{P_3(F)}^{G_{4n+1}(F)} \mu''' \otimes \pi_0,$$

where

$$\mu''' := \left( \sum_{i=1}^{n-1} \mu_i \det_i - \frac{\Omega-1}{2} f_0 \right).$$

*Proof.* Induction in stages and the definition of the unramified constituent.  $\square$

An important fact about  $\pi_0$  is the following:

**Lemma 20.0.13.** *The representation  $\pi_0$  may be realized as a subrepresentation of*

$$\text{Ind}_{R(F)}^{GSp_4(F)} \left( -\frac{1}{2} + \frac{\pi i}{\log q_F} \right) \det_0.$$

*Proof.* In fact, it is one of the spaces  $R_2(V)$  introduced on p. 223 of [K-R]. This can be checked by direct computation. It also follows from Proposition 5.5 of [K-R], in that the intertwining operator is easily seen not to vanish on the spherical vector.  $\square$

**Corollary 20.0.14.** *The representation  $\text{Ind}_{P_3(F)}^{G_{4n+1}(F)} \mu''' \otimes \pi_0$  may be realized as a subrepresentation of*

$$\text{Ind}_{P_2(F)}^{G_{4n+1}(F)} \left[ \mu''' + \left( -\frac{1}{2} + \frac{\pi i}{\log q_F} \right) \det_0 \right].$$

A second important fact about the representation  $\pi_0$  is the following:

**Lemma 20.0.15.** *Let  $\vartheta$  be a character of the unipotent radical of  $R$  in general position. Regarding  $R$  as the parabolic  $Q_1$  of  $G_5$ , the square class  $\text{Invt}(\vartheta)$  is defined. A sufficient condition for the vanishing of the twisted Jacquet module  $\mathcal{J}_{N_1, \vartheta}(\pi_0)$  is that the Hilbert symbol  $(\cdot, \text{Invt}(\vartheta))$  not equal the unique nontrivial unramified quadratic character.*

*Proof.* This follows from [K-R], Lemma 3.5 (b), p. 226. (Here, we again use the fact that the unramified constituent of  $\text{Ind}_{R(F)}^{GSp_4(F)} \left( \frac{1}{2} + \frac{\pi i}{\log q_F} \right) \det_0$  is one of the spaces  $R_2(V)$  introduced on p. 223 of [K-R].)  $\square$

**Proposition 20.0.16.** *Let  $\tau = \text{Ind}_{B(GL_{2n}(F))}^{GL_{2n}(F)} \mu$ , with  $\mu$  of the form (20.0.5), and let  $P$  denote the Siegel parabolic subgroup. Then for  $\ell > n$  and  $\vartheta$  in general position, the Jacquet module  $\mathcal{J}_{N_\ell, \vartheta}(\text{un} \text{Ind}_{P(F)}^{G_{4n+1}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega)$  is trivial. The same is true if  $\ell = n$  and  $\text{Invt}(\vartheta) \neq \square$ .*

*Proof.* By Lemma 20.0.8, it suffices to prove that the corresponding Jacquet module of  $\text{Ind}_{P_1(F)}^{G_{4n+1}(F)} \mu'$  vanishes. This is in essence an application of theorem 5.2 of [BZ2]. The space  $\text{Ind}_{P_1(F)}^{G_{4n+1}(F)} \mu'$  has a filtration as a  $Q_\ell(F)$ -module, in terms of  $Q_\ell(F)$ -modules indexed by the elements of  $(W \cap P_1) \backslash W / (W \cap Q_\ell)$ . For any element  $x$  of  $P_1(F)wQ_\ell(F)$  the module corresponding to  $w$  is isomorphic to  $c - \text{ind}_{x^{-1}P_1(F)x \cap Q_\ell(F)}^{Q_\ell(F)} (\mu' + \rho_{P_1}) \circ \text{Ad}(x)$ . Here  $\text{Ad}(x)$  denotes the map given by conjugation by  $x$ . It sends  $x^{-1}P_1(F)x \cap Q_\ell(F)$  into  $P_1(F)$ . Also, here and throughout  $c - \text{ind}$  denotes non-normalized compact induction. (See [Cass], section 6.3.)

Recall from 13.2 that the elements of the Weyl group of  $G_{4n+1}$  are (after the choice of  $\text{pr}$ ) in natural one-to-one correspondence with the set of permutations  $w \in \mathfrak{S}_{4n+1}$  satisfying,

$$(1) \ w(4n+1-i) = 4n+1-w(i)$$

As representatives for the double cosets  $(W \cap P_1) \backslash W / (W \cap Q_\ell)$  we choose the element of minimal length in each. The permutations corresponding to these elements satisfy

$$(2) \ w^{-1}(2i) > w^{-1}(2i-1) \text{ for } i = 1 \text{ to } 2n, \text{ and}$$

$$(3) \ \ell < i < j < 4n+2-\ell \implies w(i) < w(j).$$

Let  $I_w$  be the  $Q_\ell(F)$ -module obtained as

$$c - \text{ind}_{\dot{w}^{-1}P_1(F)\dot{w} \cap Q_\ell(F)}^{Q_\ell(F)} (\mu' + \rho_{P_1}) \circ \text{Ad}(\dot{w})$$

using any element  $\dot{w}$  of  $\text{pr}^{-1}(\det w \cdot w)$ . (Cf. section 13.2.)

A function  $f$  in  $I_w$  will map to zero under the natural projection to  $\mathcal{J}_{N_\ell, \vartheta}(I_w)$  iff there exists a compact subgroup  $N_\ell^0$  of  $N_\ell(F)$  such that

$$\int_{N_\ell^0} f(hn) \overline{\vartheta(n)} dn = 0 \quad \forall h \in Q_\ell(F).$$



(See [Cass], section 3.2.) Let  $h \cdot \vartheta(n) = \vartheta(h^{-1}nh)$ . It is easy to see that the integral above vanishes for suitable  $N_\ell^0$  whenever

$$(20.0.17) \quad h \cdot \vartheta|_{N_\ell(F) \cap w^{-1}P_1(F)w} \text{ is nontrivial.}$$

Furthermore, the function  $h \mapsto h \cdot \vartheta$  is continuous in  $h$ , (the topology on the space of characters of  $N_\ell(F)$  being defined by identifying it with a finite dimensional  $F$ -vector space, cf. section 5) so if this condition holds for all  $h$  in a compact set, then  $N_\ell^0$  can be made uniform in  $h$ .

Now,  $\vartheta$  is in general position. Hence, so is  $h \cdot \vartheta$  for every  $h$ . So, if we write

$$h \cdot \vartheta(u) = \psi_0(c_1 u_{1,2} + \cdots + c_{\ell-1} u_{\ell-1,\ell} + d_1 u_{\ell,\ell+1} + \cdots + d_{4n-2\ell+1} u_{\ell,4n-\ell+1}),$$

we have that  $c_i \neq 0$  for all  $i$  and  ${}^t \underline{d} J \underline{d} \neq 0$ .

Clearly, the condition (20.0.17) holds for all  $h$  unless

$$(4) \quad w(1) > w(2) > \cdots > w(\ell).$$

Furthermore, because  ${}^t \underline{d} J \underline{d} \neq 0$ , there exists some  $i_0$  with  $\ell + 1 \leq i_0 \leq 2n$  such that  $d_{i_0-\ell} \neq 0$  and  $d_{4n+2+\ell-i_0} \neq 0$ . From this we deduce that the condition (20.0.17) holds for all  $h$  unless  $w$  has the additional property

$$(5) \quad \text{There exists } i_0 \text{ such that } w(\ell) > w(i_0) \text{ and } w(\ell) > w(4n+2-i_0).$$

However, if  $\ell > n$  it is easy to check that no permutations with properties (1),(2), (4) and (5) exist.

Thus  $\mathcal{J}_{N_\ell, \vartheta}(I_w) = \{0\}$  for all  $w$  and hence  $\mathcal{J}_{N_\ell, \vartheta}({}^{un} \text{Ind}_{P(F)}^{G_{4n}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega) = \{0\}$  by exactness of the Jacquet functor.

If  $\ell = n$ , there is exactly one permutation  $w$  which satisfies (1)-(4). For this permutation, condition (4) is satisfied only with  $i_0 = 4n+2-i_0 = 2n+1$ . The orbit of  $\vartheta$  contains characters such that  $d_i = 0$  for all  $i \neq 2n+1$  iff  $\text{Invt}(\vartheta) = \square$ .  $\square$

**Proposition 20.0.18.** *Let  $\tau = \text{Ind}_{B(GL_{2n}(F))}^{GL_{2n}(F)} \mu$ , with  $\mu$  of the form (20.0.6), and let  $P$  denote the Siegel parabolic subgroup. Then for  $\ell > n$  and  $\vartheta$  in general position, the Jacquet module  $\mathcal{J}_{N_\ell, \vartheta}({}^{un} \text{Ind}_{P(F)}^{G_{4n+1}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega)$  is trivial. The same is true if  $\ell = n$  and  $\text{Invt}(\vartheta) = \square$ .*

*Proof.* For  $\ell > n$ , the proof is similar to that of Proposition 20.0.16. Using Lemma 20.0.11 in place of Lemma 20.0.8, we consider a representation induced from a character of  $P_2$  rather than  $P_1$ . The effect is that in place of condition (2) from the proof of Proposition 20.0.16, we have the condition

$$(2') \quad w^{-1}(2i-1) < w^{-1}(2i), \quad 1 \leq i < n, \quad w^{-1}(2n) < w^{-1}(2n+1).$$

The set of permutations satisfying (1),(2'),(3),(4) is again empty.

The proof of vanishing when  $\ell = n$  and  $\text{Invt}(\vartheta) = \square$  is more nuanced. In this case we use both Lemma 20.0.11 and Corollary 20.0.12, obtaining *two* filtrations of

$$\text{Ind}_{P_3(F)}^{G_{4n+1}(F)} \mu''' \otimes \pi_0 \subset \text{Ind}_{P_2(F)}^{G_{4n+1}(F)} \mu'',$$

indexed by  $(W \cap P_3) \backslash W / (W \cap Q_\ell)$  and  $(W \cap P_2) \backslash W / (W \cap Q_\ell)$ . The latter is a refinement of the former, in a manner which is compatible with the natural projection

$$(W \cap P_2) \backslash W / (W \cap Q_\ell) \rightarrow (W \cap P_3) \backslash W / (W \cap Q_\ell).$$

Let us denote the elements of the first filtration by  $I_w$ ,  $w \in (W \cap P_3) \backslash W / (W \cap Q_\ell)$ , and the elements of the second by  $I'_w$ ,  $w \in (W \cap P_2) \backslash W / (W \cap Q_\ell)$ .

Now, when  $\ell = n$  there is a unique permutation  $w_0$  satisfying (1),(2'),(3), (4),(5). It is the shortest element of the double coset containing the longest element of  $W$ . It follows that  $\mathcal{J}_{N_n, \vartheta}(I'_w)$  vanishes for every  $w \neq w_0$ , and hence that  $\mathcal{J}_{N_n, \vartheta}(I_w)$  vanishes for every  $w$  other than the shortest element of  $(W \cap P_3) \cdot w_0 \cdot (W \cap Q_n)$ , which we denote  $w'_0$ .

The permutation  $w'_0$  can be described explicitly as follows:

$$w'_0(i) = \begin{cases} 4n+2-2i & 1 \leq i \leq n-1, \\ 2n-1 & i = n, \\ 2i-2n-1 & n+1 \leq i \leq 2n-1, \ 2n+3 \leq i \leq 3n+1, \\ i & 2n \leq i \leq 2n+2, \\ 2n+3 & i = 3n+2, \\ 8n+4-2i & 3n+3 \leq i \leq 4n+1. \end{cases}$$

Furthermore, the space  $I_{w'_0}$  is equal to the subspace of  ${}^{un}\text{Ind}_{P_3(F)}^{G_{4n+1}(F)} \mu''' \otimes \pi_0$  consisting of smooth functions having support in the open double coset  $P_3(F) \cdot w'_0 \cdot Q_n(F)$ . Take such a function  $f$  and take  $N_n^0 \subset N_n(F)$ , compact. Consider the integral

$$\int_{N_n^0} f(gn) \overline{\vartheta(n)} \, dn.$$

We may assume  $g = w'_0 q$  for some  $q \in Q_n(F)$ . Then we get

$$\int_{qN_n^0 q^{-1}} f(w_0 n q) \overline{q \cdot \vartheta(n)} \, dn,$$

where  $q \cdot \vartheta(n) = \vartheta(q^{-1} n q)$ . Hence, we consider

$$(20.0.19) \quad \int_{N_n^{0'}} f'(w_0 n) \overline{\vartheta'(n)} \, dn,$$

for  $\vartheta'$  a character of  $N_n$  such that  $\text{Inv}(\vartheta') = \square$ ,  $f' \in I_{w'_0}$ , and  $N_n^{0'} \subset N_n(F)$  compact. Observe that  $w_0 N_n w_0^{-1}$  contains the unipotent radical  $U_R$  of the parabolic  $R$  of  $GS p_4$  used to define  $\pi_0$ . Indeed, if  $\hat{N}_n = \{u \in N_n : u_{n,2n} = u_{n,2n+1} = 0\}$ , then  $\hat{N}_n$  is a normal subgroup of  $N_n$  and  $N_n = w_0^{-1} U_R w_0 \cdot \hat{N}_n$ . If  $U \subset U_{\max}$ , write  $U(\mathfrak{p}^N)$  for  $\{u \in U : u_{ij} \in \mathfrak{p}^N \forall i, j\}$ .

For each  $h \in G_{4n+1}(F)$ , the function  $g \mapsto f'(gh)$ ,  $g \in GS p_4(F)$  is an element of  $\pi_0$ . By Lemma 20.0.15, for each  $h$  there exists  $N$  such that

$$\int_{w_0^{-1} U_R(\mathfrak{p}^N) w_0} f'(w_0 u h) \vartheta'(u) \, du = 0.$$

Clearly,  $N$  depends on  $f'$  and  $\vartheta'$ , and hence, if  $f'(g) = f(g \cdot q)$  and  $\vartheta' = q \cdot \vartheta$ , on  $q$ . However,  $f$  is smooth and has support which is compact modulo  $P_3(F)$ , so  $f'$  takes only finitely many values. Furthermore, the  $q \cdot \vartheta$  is a continuous function of  $q$  in the sense discussed above. Thus,  $N$  may be made uniform in  $q$ .  $\square$

Define a character  $\Psi_n$  of  $N_n(F)$  by the same formula as in Definition 16.1.8. In the proof of Lemma 16.1.10, we fixed a specific isomorphism  $\text{inc} : G_{2n} \rightarrow (L_n^{\Psi_n})^0$ . For the next proposition only, we let  $B$  denote the image under  $\text{inc}$  of the Borel  $B(G_{2n})$  corresponding to our choices of maximal torus and simple roots for  $G_{2n}$ . It is equal to  $(L_n^{\Psi_n})^0 \cap B(G_{4n+1})$ . The corresponding maximal torus  $T$  is the subtorus  $\langle e_i^* : i = 0, \text{ or } n+1 \leq i \leq 2n \rangle$ . Because of this  $\sum_{i=0}^{2n} c_i e_i$  makes sense as a character of  $T(F)$ . (But depends only on  $c_i$ ,  $i = 0, \text{ or } n+1 \leq i \leq 2n$ .)

**Proposition 20.0.20.** *Let  $P_1$ , and  $\mu'$  be defined as in Lemma 20.0.8. Then we have isomorphisms*

$$\begin{aligned} \mathcal{J}_{N_n, \Psi_n}(\text{Ind}_{P_1(F)}^{G_{4n+1}(F)} \mu') &\cong \text{Ind}_{B(F)}^{(L_n^{\Psi_n})^{(F)}} \mu^* \cong \text{Ind}_{B(F)}^{(L_n^{\Psi_n})^{(F)}} \mu^{**} & (\text{ of } L_n^{\Psi_n} - \text{ modules}), \\ \mathcal{J}_{N_n, \Psi_n}(\text{Ind}_{P_1(F)}^{G_{4n+1}(F)} \mu') &\cong \text{Ind}_{B(F)}^{(L_n^{\Psi_n})^0(F)} \mu^* \oplus \text{Ind}_{B(F)}^{(L_n^{\Psi_n})^0(F)} \mu^{**} & (\text{ of } (L_n^{\Psi_n})^0 - \text{ modules}), \end{aligned}$$

where

$$\mu^* = \sum_{i=1}^n \mu_i e_{n+i} + \Omega e_0, \quad \mu^{**} = \sum_{i=1}^{n-1} \mu_i e_{n+i} + (\Omega - \mu_n) e_{2n} + \Omega e_0.$$

*Proof.* As before, we filter  $\text{Ind}_{P_1(F)}^{G_{4n+1}(F)} \mu'$  in terms of  $Q_n(F)$ -modules  $I_w$ . This time,  $\mathcal{J}_{N_n, \Psi_n}(I_w) = \{0\}$  for all  $w$  except possibly for one. This one Weyl element, which we denote  $w_0$ , corresponds to the unique permutation satisfying (1) and (2) of Proposition 20.0.16, together with  $w_0(i) = 4n - 2i + 2$  for  $i = 1$  to  $n$ . Exactness yields

$$\mathcal{J}_{N_n, \Psi_n} \left( {}^{un}\text{Ind}_{P(F)}^{G_{4n+1}(F)} \tau \otimes |\det|^{\frac{1}{2}} \boxtimes \omega \right) \cong \mathcal{J}_{N_n, \Psi_n}(I_{w_0}).$$

(This is an isomorphism of  $Q_n^{\Psi_n}(F)$ -modules, where  $Q_n^{\Psi_n} = N_n \cdot L_n^{\Psi_n} \subset Q_n$ , is the stabilizer of  $\Psi_n$  in  $Q_n$  (cf.  $L^\vartheta$  above).)

Now, recall that for each  $h \in Q_n(F)$  the character  $h \cdot \Psi_n(u) = \Psi_n(h^{-1}uh)$  is a character of  $N_n$  in general position, and as such determines coefficients  ${}^h c_1, \dots, {}^h c_{n-1}$  and  ${}^h d_1, \dots, {}^h d_{2n+1}$  as in (16.1.3). Clearly,

$$Q_n^o := \left\{ h \in Q_n(F) \mid d_i^h \neq 0 \text{ for some } i \neq n+1, \right\}$$

is open. Moreover, one may see from the description of  $w_0$  that for  $h$  in this set the condition (20.0.17), which assures vanishing, is satisfied.

We have an exact sequence of  $Q_n^{\Psi_n}(F)$ -modules

$$0 \rightarrow I_{w_0}^* \rightarrow I_{w_0} \rightarrow \bar{I}_{w_0} \rightarrow 0,$$

where  $I_w^*$  consists of those functions in  $I_w$  whose compact support happens to be contained in  $Q_n^o$ , and the third arrow is restriction to the complement of  $Q_n^o$ . This complement is slightly larger than  $Q_n^{\Psi_n}(F)$  in that it contains the full torus of  $Q_n(F)$ , but restriction of functions is an isomorphism of  $Q_n^{\Psi_n}(F)$ -modules,

$$\bar{I}_{w_0} \rightarrow c - \text{ind}_{Q_n^{\Psi_n}(F) \cap w_0^{-1}P_1(F)w_0}^{Q_n^{\Psi_n}(F)} (\mu' + \rho_{P_1}) \circ \text{Ad}(w_0).$$

Clearly  $\mathcal{J}_{N_n, \Psi_n}(I_{w_0}^*) = \{0\}$ , and hence we have the isomorphism

$$\mathcal{J}_{N_n, \Psi_n} \left( \text{Ind}_{P_1(F)}^{G_{4n+1}(F)} \mu' \right) \cong \mathcal{J}_{N_n, \Psi_n} \left( c - \text{ind}_{Q_n^{\Psi_n}(F) \cap w_0^{-1}P_1(F)w_0}^{Q_n^{\Psi_n}(F)} (\mu' + \rho_{P_1}) \circ \text{Ad}(w_0) \right)$$

of  $Q_n^{\Psi_n}$ -modules.

Let us denote

$$c - \text{ind}_{Q_n^{\Psi_n}(F) \cap w_0^{-1}P_1(F)w_0}^{Q_n^{\Psi_n}(F)} (\mu' + \rho_{P_1}) \circ \text{Ad}(w_0)$$

by  $V$ . A straightforward computation shows that the functions in  $V$  satisfy

$$f(bq) = b^{\mu^* + \rho_B - J} f(q) \quad \forall b \in B(F), q \in Q_n^{\Psi_n}(F),$$

where

$$J = \sum_{i=1}^n (i - n - 1) e_{n+i}.$$

For  $f \in V$ , let

$$W(f)(q) = \int_{N_n(F) \cap w_0^{-1} \overline{U_{\max}(F)} w_0} f(uq) \bar{\Psi}_n(u) du.$$

Then the character  $J$  computed above matches exactly the Jacobian of  $\text{Ad}(b)$ ,  $b \in B(F)$ , acting on  $N_n(F) \cap w_0^{-1} \overline{U_{\max}(F)} w_0$ . It follows that

$$W(f)(bq) = b^{\mu^* + \rho_B} f(q) \quad \forall b \in B(F), q \in Q(F).$$

Now let  $\mathcal{W}$  denote

$$\left\{ f : Q_n^{\Psi_n}(F) \rightarrow \mathbb{C} \left| \begin{array}{l} f(uq) = \Psi_n(u)f(q) \ \forall u \in N_n(F), \ q \in Q_n^{\Psi_n}(F), \\ f(bm) = b^{\mu^* + \rho_B} f(m) \ \forall b \in B(F), \ m \in L_n^{\Psi_n}(F) \end{array} \right. \right\}.$$

Then  $\mathcal{W}$  maps  $V$  into  $\mathcal{W}$ .

Denote by  $V(N_n, \Psi_n)$  the kernel of the linear map  $V \rightarrow \mathcal{J}_{N_n, \Psi_n}(V)$ . It is easy to show that  $V(N_n, \Psi_n)$  is contained in the kernel of  $W$ . In the Lemma 20.0.22 below, we show that in fact, they are equal. Restriction from  $Q_n^{\Psi_n}(F)$  to  $L_n^{\Psi_n}(F)$  is clearly an isomorphism  $\mathcal{W} \rightarrow \text{Ind}_{B(F)}^{L_n^{\Psi_n}(F)} \mu^*$ .

The proof that this is isomorphic to  $\text{Ind}_{B(F)}^{L_n^{\Psi_n}(F)} \mu^{**}$  and decomposes into  $(L_n^{\Psi_n})^0$ -modules in the manner described is straightforward.  $\square$

The next proposition is similar. However, there is an interesting difference between the two. In the previous proposition, we let  $B$  denote the Borel subgroup  $B(G_{4n+1}) \cap (L_n^{\Psi_n})^0$  of  $(L_n^{\Psi_n})^0 \cong G_{2n}^1$ . For the next, we use it to denote  $Q_{2n-1} \cap G_{2n}^a$ , which is a Borel subgroup of  $G_{2n}^a$ . The corresponding maximal torus,  $G_{2n}^a \cap L_{2n-1}$ , is given by

$$\left\{ h_a \prod_{i=1}^{n-1} e_{n+i}^*(t_i) \cdot e_{2n}^* \left( (x + y\sqrt{a}) \cdot (x - y\sqrt{a})^{-1} \right) e_0(x - y\sqrt{a}) h_a^{-1} : t_i \in F, \ x, y \in F, x^2 - ay^2 \neq 0 \right\},$$

as in Lemma 16.1.10(3). Here  $\sqrt{a}$  may be taken to be either of the solutions to  $\zeta^2 = a$  in the algebraic closure of  $F$ . We assume  $\sqrt{a} \notin F$ . The lattice of  $F$ -rational characters of this torus is  $\langle e_{n+i} : 1 \leq i \leq n-1, e_{2n} + 2e_0 \rangle$ . The character  $e_{2n} + 2e_0$  is the restriction of a rational character of the  $L_{2n-1} \cong GL_1^{2n-1} \times GSpin_3$ . To be precise, it is the *inverse* of the character  $\det_0$  introduced earlier. (Cf. Lemma 20.0.11.) Thus, a general rational character of this torus may be expressed as

$$\sum_{i=1}^{n-1} c_i e_{n+i} + c_0 \det_0,$$

with  $c_i \in \mathbb{Z}$ . In particular map, the restriction map from  $X(L_{2n-1})$  is surjective. A general unramified character of this torus may be expressed in the same form with  $c_i \in \mathbb{C}$ . Then  $c_i$

Observe that for any  $t$  in this torus  $\det_0(t)$  is a norm from  $F(\sqrt{a})$ . When  $a$  is in the square class which contains the non-square units (i.e., when  $F(\sqrt{a})$  is the unique unramified quadratic extension of  $F$ ), the absolute value of a norm is always an even power of  $q_F$ , and so  $c_0$  is defined only up to  $\frac{\pi i}{\log q_F}$ . (whereas the others are defined up to  $\frac{2\pi i}{\log q_F}$  for  $1 \leq i \leq n-1$ .)

We also let  $\tilde{B}$  denote  $Q_{2n-1} \cap L_n^{\Psi_n^a}$ . (Recall that  $G_{2n}^a := (L_n^{\Psi_n^a})^0$ .) It is not difficult to see that  $L_{2n-1} \cap L_n^{\Psi_n^a}$  is properly larger than  $L_{2n-1} \cap (L_n^{\Psi_n^a})^0$ , i.e., contains elements of the non-identity component of  $L_n^{\Psi_n^a}$ . A character of  $B$  may be extended trivially to  $\tilde{B}$ . And any character of  $\tilde{B}$  which is obtained as the restriction of a character of  $Q_{2n-1}$  is such a trivial extension.

**Proposition 20.0.21.** *Let  $P_2$ , and  $\mu''$  be defined as in Lemma 20.0.11. Then we have isomorphisms*

$$\mathcal{J}_{N_n, \Psi_n}(\text{Ind}_{P_2(F)}^{G_{4n+1}(F)} \mu'') \cong \text{Ind}_{\tilde{B}(F)}^{(L_n^{\Psi_n^a})^0(F)} \mu^* \quad (\text{ of } L_n^{\Psi_n^a} - \text{modules}),$$

$$\mathcal{J}_{N_n, \Psi_n}(\text{Ind}_{P_2(F)}^{G_{4n+1}(F)} \mu'') \cong \text{Ind}_{B(F)}^{G_{2n}^a(F)} \mu^* \quad (\text{ of } (L_n^{\Psi_n})^0 - \text{modules}),$$

where

$$\mu^* = \sum_{i=1}^{n-1} \mu_i e_{n+i} - \left( \frac{\Omega}{2} + \frac{\pi i}{\log q_F} \right) \det_0.$$

*Proof.* We use Lemma 20.0.11, and filter by  $Q_n$ -modules. As in Proposition 20.0.20, there is a unique permutation  $w_1$  such that the corresponding  $Q_n$ -module  $I_{w_1}$  does not vanish. This permutation is given by

$$w_1(i) = \begin{cases} 4n+2-2i & 1 \leq i \leq n-1, \\ 2n+3 & i = n, \\ 2i-2n-1 & n+1 \leq i \leq 2n-1, \\ i & 2n \leq i \leq 2n+2, \\ 2i-2n-1 & 2n+3 \leq i \leq 3n+1, \\ 2n-1 & i = 3n+2, \\ 2(4n+2-i) & 3n+3 \leq i \leq 4n+1. \end{cases}$$

The group  $Q_n \cap w_1^{-1}P_2w_1$  contains  $L_{2n-1}$ . Since  $L_{2n-1} \cdot Q_n^{\Psi_n^a} = Q_n$ , restriction of functions is an isomorphism of  $Q_n^{\Psi_n^a}$ -modules,

$$I_{w_1} \rightarrow c - \text{ind}_{Q_n^{\Psi_n^a} \cap w_1^{-1}P_2w_1}^{Q_n^{\Psi_n^a}} (\mu'' + \rho_{P_2}) \circ \text{Ad}(w_1).$$

This time, let  $V$  denote

$$c - \text{ind}_{Q_n^{\Psi_n^a} \cap w_1^{-1}P_2w_1}^{Q_n^{\Psi_n^a}} (\mu'' + \rho_{P_2}) \circ \text{Ad}(w_1).$$

Once again the functions in  $V$  satisfy

$$f(bq) = b^{\mu^* + \rho_B - J} f(q) \quad \forall b \in B(F), q \in Q_n^{\Psi_n}(F),$$

with  $J$  as before. We define

$$W(f)(q) = \int_{N_n(F) \cap w_1^{-1}\overline{U_{\max}(F)}w_1} f(uq) \bar{\Psi}_n(u) du,$$

and find that  $W$  maps  $V$  to

$$\mathcal{W} := \left\{ f : Q_n^{\Psi_n}(F) \rightarrow \mathbb{C} \left| \begin{array}{l} f(uq) = \Psi_n(u) f(q) \quad \forall u \in N_n(F), q \in Q_n^{\Psi_n}(F), \\ f(bm) = b^{\mu^* + \rho_B} f(m) \quad \forall b \in B(F), m \in L_n^{\Psi_n}(F) \end{array} \right. \right\},$$

which is easily seen to be isomorphic to each of the induced representations specified. As before, the kernel of the linear map  $V \rightarrow \mathcal{J}_{N_n, \Psi_n}(V)$  is contained in the kernel of  $W$ . In Lemma 20.0.22, we show that in fact, they are equal to complete the proof.  $\square$

**Lemma 20.0.22.** *Let  $\vartheta$  be a character of  $N_n$  in general position,  $H$  its stabilizer in  $L_n$ ,  $U_1$  and  $U_2$  two subgroups of  $N_n$  such that  $U_1 \cap U_2 = 1$  and  $U_1 U_2 = U_2 U_1 = N_n$ . Let  $B$  denote a Borel subgroup of the identity component of  $H$  and  $\chi$  a character of  $B$ . Assume*

$$(20.0.23) \quad B(F)H(\mathfrak{o}) = H(F).$$

*Let  $V$  denote a space of functions on  $N_n(F) \cdot H(F)$  which are compactly supported modulo  $U_1(F)$  on the left and satisfy*

$$f(u_1 b q) = \chi(b) f(q) \quad \forall u_1 \in U_1(F), b \in B(F), q \in H(F)N_n(F).$$

*Let  $V(N_n, \vartheta)$  denote the kernel of the usual projection from  $V$  to its twisted Jacquet module.*

*Let*

$$W(f)(q) = \int_{U_2(F)} f(u_2 q) \bar{\vartheta}(u_2) du_2.$$

*Then  $\text{Ker}(W) \subset V(N_n, \vartheta)$ .*

*Proof.* We assume that

$$\int_{U_2(F)} f(uq)\bar{\vartheta}(u)du = 0,$$

for all  $q \in H(F)N_n(F)$ . What must be shown is that there is a compact subset  $C$  of  $N_n(F)$  such that

$$\int_C f(gu)\bar{\vartheta}(u)du = 0,$$

for all  $q \in H(F)N_n(F)$ .

Consider first  $m \in H(\mathfrak{o})$ . Let  $\mathfrak{p}$  denote the unique maximal ideal in  $\mathfrak{o}$ . If  $U$  is a unipotent subgroup and  $M$  an integer, we define

$$U(\mathfrak{p}^M) = \{u \in U(F) : u_{ij} \in \mathfrak{p}^M \ \forall i \neq j\}.$$

Observe that for each  $M \in \mathbb{N}$ ,  $N_n(\mathfrak{p}^M)$  is a subgroup of  $N_n(F)$  which is preserved by conjugation by elements of  $H(\mathfrak{o})$ . We may choose  $M$  sufficiently large that  $\text{supp}(f) \subset U_1(F)U_2(\mathfrak{p}^{-M})H(F)$ . Then we prove the desired assertion with  $C = N_n(\mathfrak{p}^{-M})$ . Indeed, for  $m \in H(\mathfrak{o})$ , we have

$$\int_{N_n(\mathfrak{p}^{-M})} f(mu)\bar{\vartheta}(u)du = \int_{N_n(\mathfrak{p}^{-M})} f(um)\bar{\vartheta}(u)du,$$

because  $Ad(m)$  preserves the subgroup  $N_n(\mathfrak{p}^{-M})$ , and has Jacobian 1. Let  $c = \text{Vol}(U_1(\mathfrak{p}^{-M}))$ , which is finite. Then by  $U_1$ -invariance of  $f$ , the above equals

$$= c \int_{U_2(\mathfrak{p}^{-M})} f(um)\bar{\vartheta}(u)du.$$

This, in turn, is equal to

$$= c \int_{U_2(F)} f(um)\bar{\vartheta}(u)du,$$

since none of the points we have added to the domain of integration are in the support of  $f$ , and this last integral is equal to zero by hypothesis.

Next, suppose  $q = u_2m$  with  $u_2 \in U_2(F)$  and  $m \in H(\mathfrak{o})$ . If  $u_2 \in U_2(F) - U_2(\mathfrak{p}^{-M})$  then  $qu$  is not in the support of  $f$  for any  $u \in U_2(\mathfrak{p}^{-M})$ . On the other hand, if  $u_2 \in U_2(\mathfrak{p}^{-M})$ , then

$$\begin{aligned} \int_{N_n(\mathfrak{p}^{-M})} f(u_2mu)\bar{\vartheta}(u)du &= \int_{N_n(\mathfrak{p}^{-M})} f(u_2um)\bar{\vartheta}(u)du \\ &= \vartheta(u_2) \int_{N_n(\mathfrak{p}^{-M})} f(um)\bar{\vartheta}(u)du, \end{aligned}$$

and now we continue as in the case  $u_1 = 1$ .

The result for general  $q$  now follows from the left-equivariance properties of  $f$  and (20.0.23).  $\square$

## 21. APPENDIX VII: IDENTITIES OF UNIPOTENT PERIODS

**Lemma 21.0.24.** *Let  $(U_1^a, \psi_1^a)$  and  $(U_2, \psi_2^a)$  be defined as in Theorem 16.3.1. Then  $(U_1^a, \psi_1^a) \sim (U_2, \psi_2^a)$ , for all  $a \in F$ .*

*Proof.* We regard  $a$  as fixed and omit it from the notation. We define some additional unipotent periods which appear at intermediate stages in the argument. Let  $U_4$  be the subgroup defined by  $u_{n,j} = 0$  for  $j = n+1$  to  $2n-1$  and  $u_{2n,2n+1} = 0$ . We define a character  $\psi_4$  of  $U_4$  by the same formula as  $\psi_1$ . Then  $(U_1, \psi_1)$  may be swapped for  $(U_4, \psi_4)$ . (See definition 5.1.3.)



Now, for each  $k$  from 1 to  $n$ , define  $(U_5^{(k)}, \psi_5^{(k)})$  as follows. First, for each  $k$ , the group  $U_5^{(k)}$  is contained in the subgroup of  $U_{\max}$  defined by,  $u_{2n,2n+1} = 0$ . In addition,  $u_{n+k-1,j} = 0$  for  $j < 2n$ , and  $u_{i,i+1} = 0$  if  $n - k + 1 \leq i < n + k - 1$  and  $i \equiv n - k + 1 \pmod{2}$ , and

$$\psi_5^{(k)}(u) = \psi_0 \left( \sum_{i=1}^{n-k} u_{i,i+1} + \sum_{i=n-k+1}^{n+k-2} u_{i,i+2} + u_{n+k-1,2n} + \frac{a}{2} u_{n+k-1,2n+2} + \sum_{i=n+k}^{2n-2} u_{i,i+1} + u_{2n-1,2n+2} \right).$$

(Note that one or more of the sums here may be empty.)

Next, let  $U_6^{(k)}$  be the subgroup of  $U_{\max}$  defined by the conditions  $u_{2n,2n+1} = 0$ ,  $u_{n+k-1,j} = 0$  for  $j < 2n$ , and  $u_{i,i+1} = 0$  if  $n - k + 1 \leq i < n + k - 1$  and  $i \equiv n - k \pmod{2}$ . The same formula which defines  $\psi_5^{(k)}$  also defines a character of  $U_6^{(k)}$ . We denote this character by  $\psi_6^{(k)}$ .

We make the following observations:

- $(U_5^{(1)}, \psi_5^{(1)})$  is precisely  $(U_4, \psi_4)$ .
- For each  $k$ ,  $(U_5^{(k)}, \psi_5^{(k)})$  is conjugate to  $(U_6^{(k+1)}, \psi_6^{(k+1)})$ . The conjugation is accomplished by any preimage of the permutation matrix which transposes  $i$  and  $i + 1$  for  $n - k \leq i < n + k$  and  $i \equiv n - k + 1 \pmod{2}$ .
- $(U_6^{(k)}, \psi_6^{(k)})$  may be swapped for  $(U_5^{(k)}, \psi_5^{(k)})$ .

Thus  $(U_4, \psi_4) \sim (U_5^{(n-1)}, \psi_5^{(n-1)})$ .

Now, let  $U'_2 = U_5^{(n-1)}$ , and let

$$\psi'_2(u) = \psi_0(u_{1,3} + \cdots + u_{2n-2,2n} + u_{2n-2,2n+1} + \frac{a}{2} u_{2n-1,2n} + u_{2n-1,2n+2}).$$

Then  $(U_5^{(n-1)}, \psi_5^{(n-1)})$  is conjugate to  $(U'_2, \psi'_2)$ , which may be swapped for  $(U_2, \psi_2)$ .  $\square$

**Lemma 21.0.25.** *Let  $(U_3, \psi_3)$  and  $(U_2, \psi_2^0)$  be defined as in Theorem 16.3.1. Then*

$$(U_3, \psi_3) \in \langle (U_2, \psi_2^0), \{(N_\ell, \vartheta) : n \leq \ell < 2n \text{ and } \vartheta \text{ in general position.}\} \rangle.$$

*Proof.* To prove this assertion we introduce some additional unipotent periods. For  $k = n$  to  $2n - 1$  let  $U_7^{(k)}$  denote the subgroup of  $U_{\max}$  defined by  $u_{2n,2n+1} = 0$ , and  $u_{i,2n} = 0$  for  $k + 1 \leq i \leq 2n - 1$ , and let

$$\psi_7^{(k)}(u) = \psi_0 \left( \sum_{i=1}^{k-1} u_{i,i+1} + u_{k,2n} + \sum_{i=k+1}^{2n-2} u_{i,i+1} + u_{2n-1,2n+2} \right).$$

Let  $U_8^{(k)}$  denote the subgroup defined by  $u_{2n-1,2n+1} = 0$ ,  $u_{k,j} = 0$  for  $k + 1 \leq j < 2n$ , and let  $U_9^{(k)}$  denote the subgroup defined by the additional condition  $u_{k,2n} = 0$ . The same formula which defines  $\psi_7^{(k)}$  may be used to specify a character of  $U_8^{(k)}$ , which we denote  $\psi_8^{(k)}$ . In addition, let

$$\psi_0 \left( \sum_{i=1}^{k-1} u_{i,i+1} + u_{k,2n+2} + \sum_{i=k+1}^{2n-1} u_{i,i+1} \right),$$

be denoted by  $\tilde{\psi}_8^{(k)}$  for  $u \in U_8^{(k)}$  or  $\psi_9^{(k)}$  for  $u \in U_9^{(k)}$ .

Now, we need the following observations:

- $(U_7^{(n)}, \psi_7^{(n)})$  is just the period  $(U_1^0, \psi_1^0)$  from theorem 16.3.1, and so is equivalent to  $(U_2, \psi_2^0)$  by the previous result.
- For each  $k$ ,  $(U_7^{(k)}, \psi_7^{(k)})$  is conjugate to  $(U_9^{(k+1)}, \psi_9^{(k+1)})$ . (One conjugates by a preimage of a permutation matrix and then by a toral element to fix a minus sign which is introduced.)

- $(U_8^{(k+1)}, \tilde{\psi}_8^{(k+1)})$  is spanned by  $(U_9^{(k+1)}, \psi_9^{(k+1)})$  and  $\{(N_k, \vartheta) : \vartheta \text{ in general position}\}$ . More precisely, if  $\vartheta$  is any extension of  $\psi_9^{(k+1)}$  which is *not* in general position, then the restriction of  $\vartheta$  to  $U_8$  is  $\tilde{\psi}_8^{(k+1)}$ . (Cf. Corollary 5.1.2.)
- $(U_8^{(k)}, \tilde{\psi}_8^{(k)})$  is conjugate to  $(U_8^{(k)}, \psi_8^{(k)})$ .
- $(U_8^{(k)}, \psi_8^{(k)})$  may be swapped for  $(U_7^{(k)}, \psi_7^{(k)})$ .

We deduce that  $(U_2, \psi_2^0)$  divides  $(U_8^{(2n-1)}, \psi_8^{(2n-1)})$ , a period which differs from  $(U_3, \psi_3)$  only in that integration over  $u_{2n, 2n+1}$  is omitted. Thus  $(U_3, \psi_3)$  is the constant term in the Fourier expansion of  $(U_8^{(2n-1)}, \psi_8^{(2n-1)})$ , in the variable  $u_{2n, 2n+1}$ , while all of the nonconstant terms are Whittaker integrals with respect to various generic characters of  $U_{\max}$ . As  $\mathcal{E}_{-1}(\tau, \omega)$  is non-generic, they all vanish. The result follows.  $\square$

**Lemma 21.0.26.** *Take  $a \in F^\times$ . We regard  $a$  as fixed throughout and, for the most part we suppress it from the notation. As in Theorem 16.3.1, let  $V_i$  denote the unipotent radical of the standard parabolic of  $G_{4n+1}$  having Levi isomorphic to  $GL_i \times G_{4n-2i+1}$  (for  $1 \leq i \leq 2n$ ). For  $1 \leq j < 2n$ , let  $V_i^{4n-2j}$  denote the unipotent radical of the standard maximal parabolic of  $G_{4n-2j}^a$  having Levi isomorphic to  $GL_i \times G_{4n-2j-2i}^a$  (for  $1 \leq i \leq 2n-j-2$  in the split case and  $1 \leq i \leq 2n-j-2$  in the nonsplit cases). Let  $(N_\ell, \Psi_\ell^a)$  be the period used to define the descent, as usual, and let  $(N_\ell, \Psi_\ell^a)^{(4n-2k+1)}$  denote the analogue for  $G_{4n-2k+1}$ , embedded into  $G_{4n+1}$  inside the Levi of a maximal parabolic.*

*Then,  $(V_k^{2n}, \mathbf{1}) \circ (N_n, \Psi_n)$  is an element of*

$$\langle (N_{n+k}, \Psi_{n+k}), \{(N_{n+j}, \Psi_{n+j})^{(4n-2k+2j)} \circ (V_{k-j}, \mathbf{1}) : 1 \leq j < k\} \rangle.$$

*Proof.* Let  $m = (m_1, m_2, m_3)$  be a triple of integers satisfying:  $0 \leq m_1 < m_2 \leq m_3 + 1 \leq 2n$ . We associate to this data a unipotent group  $U_m$  and two characters  $\psi_m, \psi'_m$  as follows:

- $U_m$  is defined by the condition that  $u_{i,j} = 0$  whenever  $m_1 < i < m_2 - 1$  and  $j < m_2$ , or  $m_3 < i$ ,
- $\psi_m(u) = \psi_0 \left( \sum_{i=1}^{m_1} u_{i,i+1} + u_{m_1+1, m_2} + \sum_{i=m_2}^{m_3-1} u_{i,i+1} + u_{m_3, 2n} + \frac{a}{2} u_{m_3, 2n+2} \right),$
- $\psi'_m(u) = \psi_0 \left( \sum_{i=1}^{m_1-1} u_{i,i+1} + u_{m_1, m_2-1} + \sum_{i=m_2-1}^{m_3-1} u_{i,i+1} + u_{m_3, 2n} + \frac{a}{2} u_{m_3, 2n+2} \right).$

Then  $(U_m, \psi'_m)$  is conjugate to  $(U_m, \psi_m)$  and may be swapped for  $(U_{m'}, \psi_{m'})$ , where  $(m_1, m_2, m_3)' = (m_1 - 1, m_2 - 1, m_3)$ . Furthermore, for any  $k < n$ ,  $(V_k^{2n}, \mathbf{1}) \circ (N_n, \Psi_n)$  is an integral over the subgroup of  $U_{n, n+k+1, n+k}$  defined by the conditions,  $u_{i, 2n} = -\frac{a}{2} u_{i, 2n+2}$ , for  $n < i \leq n+k$ . It may be swapped for the period  $(U_m, \psi'')$  corresponding to  $m = (n-1, n+k+1, n+k)$ , and

$$\psi''(u) = \psi_0 \left( \sum_{i=1}^{n-1} u_{i,i+1} + u_{n, 2n} + \frac{a}{2} u_{n, 2n+2} \right),$$

and this period is conjugate to  $(U_m, \psi'_m)$  for this value of  $m$ . It follows that  $(V_k^{2n}, \mathbf{1}) \circ (N_n, \Psi_n)$  is equivalent to  $(U_m, \psi'_m)$  for the triple  $m = (0, k+2, n+k)$ .

Now, it's easy to see that  $(U_{(0,1,m_3)}, \psi'_{(0,1,m_3)}) = (N_{m_3}, \Psi_{m_3}^a)$ , and that for  $m_2 > 2$  there are two orbits of extensions of  $\psi_{(0,m_2,m_3)}$  to  $U_{(0,m_2-1,m_3)}$ , namely, the one containing  $\psi'_{(0,m_2-1,m_3)}$ , and the trivial extension, which yields the period  $(N_{m_3-m_2+2}, \psi_{m_3-m_2+2}^a)^{(4n-2m_2+5)} \circ (V_{m_2-2}, \mathbf{1})$ . This proves the assertions regarding all cases except for the two parabolics with Levi isomorphic to  $GL_1 \times GL_n$  in the split case.

As noted previously, it is enough to consider one of them, because they are conjugate in  $G_{4n+1}$ . Furthermore, we may conjugate by  $h_a$ , and use the more convenient embedding of  $G_{2n}^\square$  into  $G_{4n+1}$  as  $(L_n^{\Psi_n})^0$ .

For this case we take  $m \in \mathbb{Z}$  with  $0 \leq m \leq n$ , and define  $U_m$  to be the subgroup of  $U_{\max}$  defined by  $u_{i,j} = 0$  whenever  $m < i < j \leq m+n+1$ . Take

$$\begin{aligned}\psi'_m(u) &= \psi_0 \left( \sum_{i=1}^{m-1} u_{i,i+1} + u_{m,m+n+1} + \sum_{i=m+n+2}^{2n} u_{i,i+1} \right), \\ \psi''_m(u) &= \psi_0 \left( \sum_{i=1}^m u_{i,i+1} + u_{m+1,m+n+2} + \sum_{i=m+n+3}^{2n} u_{i,i+1} \right).\end{aligned}$$

Then  $(V_n^{2n}, \mathbf{1}) \circ (N_n, \Psi_n) = (U_n, \psi'_n)$ . Furthermore  $(U_m, \psi'_m)$  is conjugate to  $(U_m, \psi''_m)$  and may be swapped for  $(U_{m-1}, \psi''_{m-1})$ . Furthermore,  $(U_0, \psi'_0)$  is easily seen to be in the span of the periods

$$(U_{\max}^{4n-2k+1}, \vartheta) \circ (V_k, \mathbf{1})$$

for  $0 \leq k < n$  and  $\vartheta$  a generic character of the maximal unipotent subgroup of  $G_{4n-2k+1}$  (embedded into  $G_{4n+1}$ ) as a component of a standard Levi as usual. This completes the proof.  $\square$

So far, we have proved relations of two forms

- Equivalencies, in which the unipotent subgroup  $U$  is replaced by another of the same dimension, and the character  $\psi$  by another in the same orbit.
- Relations where one replaces  $U$  by a group of properly larger dimension, and considers all orbits of extensions of  $\psi$ .

The statement that  $(U_2, \psi_2^0)$  is spanned by  $\{(U_2, \psi_2^a) : a \in F^\times\}$  is of a different nature, and requires the use of theta functions, as in section 5.2.1.

**Lemma 21.0.27.** *Let the group  $U_2$ , and the character  $\psi_2^a$  for each  $a \in F$  be defined as in the main theorem. Then  $(U_2, \psi_2^0) \in \langle \{(U_2, \psi_2^a) : a \in F^\times\} \rangle$ .*

*Proof.* The  $R = LN$  be the unique standard parabolic subgroup of  $GSpin_{4n+1}$  such that the Levi,  $L$  is isomorphic to  $GL_2^{n-1} \times GSpin_5$ . Define a character  $\psi_N$  of the unipotent radical  $N$  by

$$\psi_N(u) = \psi_0 \left( \sum_{i=1}^{2n-2} u_{i,i+2} \right).$$

Let  $\text{Stab}_L(\psi_N)$  denote the stabilizer of  $\psi_N$  in  $L$ . Then  $\text{Stab}_L(\psi_N)$  is equal to the product of a reductive group isomorphic to  $GL_2 \times GL_1$  and three dimensional unipotent group. The image in  $SO_{4n+1}$  consists of matrices of the form

$$\text{diag}(g, g, \dots, g, g', {}_t g^{-1} \dots, {}_t g^{-1}), \quad g \in GL_2, \quad g' = \begin{pmatrix} g & * & * \\ & 1 & * \\ & & {}_t g^{-1} \end{pmatrix} \in SO_5.$$

In particular,  $\text{Stab}_L(\psi_N)$  maps isomorphically onto the Siegel parabolic of  $GSpin_5$ , which is to say the Klingen parabolic of  $GSp_4$ . This group has a subgroup which was identified with  $G^J$  above. Now

$$\varphi^{(U_2, \psi_2^a)} = \left( \varphi^{(N, \psi_N)} \right)^{(U_{\text{Si}}, \psi_{U_{\text{Si}}, a})},$$

so this result follows from corollary 5.2.6.  $\square$

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